XVIII. *On the Mathematical Theory of Electromagnetism.*

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I. Introduction.

A *Electromagnetic Coordinates.*

1. It has been thought advisable to reserve an account of the general aims and scope of the following paper till a few preliminary matters have been disposed of.

2. Consider the following statement, of the truth of which probably no one will doubt. If a body on being moved from a position A to a position B were found thereby to have lost a charge of electricity, physicists would not be content to explain the circumstance on the mere ground that it had left its charge behind. They would hold that processes had gone on, precisely similar to such as would have been required to divest it of its charge, had it remained in its first position A.

This has an important bearing on the way in which the "electric displacement" is related to matter. The polarisation thus called is some sort of *polarisation of matter, and this polarisation is carried about by the matter when it moves.* There certainly is no lack of evidence that electric actions go on in space where there is, to the best of our knowledge, no matter. In this space, however, is a medium of some sort, which is intimately related to matter, and certainly affected in some way by the motion of matter. For the present we must, for the sake of simplicity, be content to assume that the strains of this medium are, if it only bounds matter, continuous with those of matter, and if it permeates matter, are at places common to both matter and the medium identical with those of matter. (This may or may not be true. I only say that in the first development of the theory of this paper it must for simplicity be assumed.) This will not prevent us from regarding the slipping of the one medium over the other as the limit of a rapid shear. With this assumption the medium in question will appear in our equations merely as matter with zero density, but other physical quantities not zero.

Both for the medium referred to, and for matter, the statement would seem to remain true that the polarisation called electric displacement is a property that is carried about by the medium experiencing it.

3. In choosing the coordinates of any mechanical system it is, of course, only *necessary* to take them so that when given they completely specify the instantaneous position of the system. But as in all ordinary dynamical problems, so in the general electromagnetic problem, there may be all the difference in the world between one set and another in respect to the simplicity of the investigations in which they are employed, and the amount of light they throw on the interdependence of the parts of the system.

Now, I believe I am right in saying that all writers on the present subject take as the electric coordinates the three coordinates of the vector $\mathbf{D}d\varsigma$* for every element of volume $d\varsigma$ in space where $\mathbf{D}$ is the electric displacement at the point. According to the view just advanced that $\mathbf{D}$ is the measure of a property of the *matter* occupying the element $d\varsigma$, which is carried about with the matter, these are unsuitable coordinates. According to that view it is probable that the electric current is as intimately connected with the matter in which it resides, as is the electric displacement. It would seem to follow that the current components could not in general be considered as the rates of variation of the corresponding electric coordinates.

4. Suppose all space split up into a series of elementary parallelepipeda which *move with matter*. Let $\pm d\Sigma_a'$, $\pm d\Sigma_b'$, $\pm d\Sigma_c'$ be the six vector faces of one such parallelepiped. We shall take for our electric coordinates the three quantities $S\mathbf{D}'d\Sigma_a'$, $S\mathbf{D}'d\Sigma_b'$, $S\mathbf{D}'d\Sigma_c'$, where $\mathbf{D}'$ is the electric displacement at the point, for every element in space. [The reason for the dashes will appear immediately.]

Moreover, we assume that the same expressions, when $\mathbf{D}'$ is replaced by $\mathbf{C}'$, the current, are the rates of variation of the corresponding coordinates. In other words, the current $\mathbf{C}'$ at any point is defined by the equation

$$S\mathbf{C}'\,d\Sigma' = \frac{dS\mathbf{D}'d\Sigma'}{dt} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (1),$$

where $d\Sigma'$ is any vector element of surface which *moves with matter*, and d/dt denote differentiation with regard to the time which follows the motion of matter. Thus the whole current through any surface which moves with matter = the rate of variation of the whole displacement through that surface.

B. *Mathematical Machinery.*

5. As might be expected, the mathematical machinery that appears to be most convenient for investigating as fully as possible the consequences of these assumptions, and others intimately connected with them, is novel. And I may remark in passing that what Professor TAIT persistently and with complete justice emphasizes as one of the greatest boons that Quaternions grant to ungrateful physicists, viz.,

* Throughout this paper Mr. HEAVISIDE's practice of replacing MAXWELL's German letters by thick ordinary type is followed.

their *perfect naturalness*, seems to me to receive illustration in the methods about to be described.

The notation in the present paper will be mainly the same as that of my former paper on a "Proposed Extension of the Powers of Quaternion Differentiation."*

As in that paper (which will for the future be referred to as "the former paper"), a fixed position of all the matter in space will be taken as a standard of reference. Most of the following symbols have exactly the same meaning as before.

ρ is the coordinate vector of any particle of matter in the standard position ; ρ' the coordinate of the same particle in the present position, so that ρ' may be regarded as a function of the independent variables, t (the time), and ρ. $d\varsigma$, $d\varsigma'$ denote elements of volume of the same particle in the standard and present positions; ds, ds' similar elements of surface; and $U\nu$, $U\nu'$ the unit normals at ds, ds'. In the present paper another notation will also be used, defined by

$$d\Sigma = U\nu\, ds,\ d\Sigma' = U\nu'\, ds' \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (1),$$

whence

$$U\nu = U\, d\Sigma,\ ds = T\, d\Sigma \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (2),$$

and similarly for $U\, d\Sigma'$, $T\, d\Sigma'$. This meaning of Σ is scarcely likely to clash with the usual summation meaning (which will also be freely used in the present paper), since in the present use the Σ will always be preceded by d, a combination that would be rare with the ordinary meaning.

With this notation equations (2) and (3) of the former paper take the somewhat briefer form

$$\int \phi\, d\rho = \iint \phi \nabla\, d\Sigma\, \Delta \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (3),$$

$$\iint \phi\, d\Sigma = \iiint \phi \Delta\, d\varsigma \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (4).$$

In connection with these equations it is well to call attention to the following usual conventions which will be strictly adhered to. The right-handed system of rotation is adopted. $U\nu$, or $d\Sigma$, when regarded, as in the last equation, as the normal of the boundary of any region, is always drawn *from the region bounded.* Thus, if $U\nu$ is regarded as the normal to the boundary of a dielectric at its junction with a conductor, it is drawn from the point of the bounding surface into the conductor. The positive direction, that of $d\rho$ in equation (3), round the boundary of a surface, is that of positive rotation round a proximate positive normal, $d\Sigma$ in equation (3). Thus the positive direction round the boundary of a magnetic shell whose positive normal is in the direction of magnetisation is that of the equivalent current.

∇ will have the usual meaning with regard to ρ, and ∇' the same meaning with

* 'Proceedings of the Royal Society of Edinburgh,' 1890–91, p. 98.

regard to ρ'. Δ, a particular form of ∇, is used when we wish to imply that the differentiations of the ∇ are to refer to *all* the factors of a term. Thus

$$\mathrm{V}\,\mathbf{D}\Delta\mathbf{E} = \Sigma\,\partial\,(\mathrm{V}\mathbf{D}i\mathbf{E})/\partial x.$$

If σ be an independent variable vector, ${}_\sigma\nabla$, ${}_\sigma\Delta$, have the same meanings with regard to σ as ∇, Δ, with regard to ρ.

Cl is a symbol of differentiation which is thus defined:—if ϖ be an independent variable self-conjugate linear vector function of a vector, given in terms of the scalars P and c by means of the equations

$$\varpi i = \mathrm{P}i + \mathrm{N}j + \mathrm{M}k$$
$$\varpi j = \mathrm{N}i + \mathrm{Q}j + \mathrm{L}k$$
$$\varpi k = \mathrm{M}i + \mathrm{L}j + \mathrm{R}k,$$

${}_\varpi\mathrm{Cl}$ is a symbolic self-conjugate linear vector function of a vector given by

$$2{}_\varpi\mathrm{Cl}\,i = 2i\frac{\partial}{\partial\mathrm{P}} + j\frac{\partial}{\partial\mathrm{N}} + k\frac{\partial}{\partial\mathrm{M}}$$
$$2{}_\varpi\mathrm{Cl}\,j = i\frac{\partial}{\partial\mathrm{N}} + 2j\frac{\partial}{\partial\mathrm{Q}} + k\frac{\partial}{\partial\mathrm{L}}$$
$$2{}_\varpi\mathrm{Cl}\,k = i\frac{\partial}{\partial\mathrm{M}} + j\frac{\partial}{\partial\mathrm{L}} + 2k\frac{\partial}{\partial\mathrm{R}}.$$

Numerical suffixes are used exclusively to denote to what symbols the differentiations of a ∇ or Cl refer, the operator and the operand having for this purpose the same suffix.

Let Q (α, β) be any function of two independent vectors α, β, which is linear in each Then ζ is defined by the equation

$$\mathrm{Q}\,(\zeta, \zeta) = \mathrm{Q}\,(\nabla_1, \rho_1) = \mathrm{Q}\,(i, i) + \mathrm{Q}(j, j) + \mathrm{Q}\,(k, k).$$

Similarly if R $(\alpha, \beta, \gamma, \delta)$ be linear in each of its constituents

$$\mathrm{R}\,(\zeta_1, \zeta_1, \zeta_2, \zeta_2) = \mathrm{R}\,(\nabla_1, \rho_1, \nabla_2, \rho_2),$$

and so to any number of pairs of ζ's.

At a given instant ρ' is a function of ρ only, and, therefore,

$$d\rho' = -\,\mathrm{S}\,d\rho\nabla\,.\,\rho' = \chi\,d\rho,$$

where χ is a linear vector function which is called the strain function. q, ψ, Ψ, m are all functions of χ given by the equations

$$\chi w = q\psi w q^{-1},$$

where q is a quaternion and ψ a *self-conjugate* linear vector function of a vector. χ' being the conjugate of χ,

$$\chi'\chi = \psi^2 = \Psi,$$

$$m = \frac{\mathrm{S}\,d\rho_a'\,d\rho_b'\,d\rho_c'}{\mathrm{S}\,d\rho_a\,d\rho_b\,d\rho_c} = \frac{d\varsigma'}{d\varsigma},$$

where $d\rho_a$, $d\rho_b$, $d\rho_c$ are three arbitrary independent increments of ρ, and $d\rho_a'$, $d\rho_b'$, $d\rho_c'$ the consequent increments of ρ'.

F and ϕ will have meanings closely connected but not identical with their meanings in the former paper. This will be explained later.

6. The displacement, current, magnetic force, &c., at the point ρ' will not be denoted by **D**, **C**, **H**, &c., but by **D**$'$, **C**$'$, **H**$'$, &c., with which the former symbols are connected in a way now to be described. In MAXWELL'S 'Elect. and Mag.,' 2nd ed., § 12, he* remarks: "Physical vector quantities may be divided into two classes, in one of which the quantity is defined with reference to a line, while in the other the quantity is defined with reference to an area. . . . In electrical science, electromotive and magnetic intensity belong to the first class, being defined with reference to lines. When we wish to indicate this fact we may refer to them as intensities. On the other hand, electric and magnetic induction, and electric currents, belong to the second class, being defined with reference to areas. When we wish to indicate this fact we shall refer to them as fluxes." Now in connecting dashed with undashed letters it is absolutely necessary to bear in mind whether the vectors indicated are intensities or fluxes. The connection between **D** and **D**$'$ will differ from that between **H** and **H**$'$.

7. Nearly all the physical vectors at a point will belong then to one of the following classes :—

Class I. Intensities.

(Examples: ∇, **A**, **E**, **H**, Θ, $d\Sigma/d\varsigma$, ${}_\tau\nabla l$.)

σ being a vector of this class, the three allied vectors, σ, σ', σ'', are connected by the equations

$$\mathrm{S}d\rho\sigma = \mathrm{S}d\rho'\sigma',\ \ \sigma' = \chi'^{-1}\sigma,\ \ \sigma'' = q^{-1}\sigma' q = \psi^{-1}\sigma \quad . \quad . \quad . \quad . \quad (5).$$

Class II. Fluxes.

(Examples: **B**, **C**, **D**, $d\rho/d\varsigma$, ${}_\sigma\nabla l$.)

τ being a vector of this class, the three allied vectors, τ, τ', τ'', are connected by the equations

$$\mathrm{S}d\Sigma\tau = \mathrm{S}d\Sigma'\tau',\ \ \tau' = m^{-1}\chi\tau,\ \ \tau'' = q^{-1}\tau' q = m^{-1}\psi\tau \quad . \quad . \quad . \quad . \quad (6).$$

* This part of the present paper should be read in connection with MAXWELL'S paper "On the Mathematical Classification of Physical Quantities," 'Collected Scientific Papers,' vol. 2, p. 257, or 'Proc. London Math. Soc.,' vol. 3, No. 34. In connection with the naturalness of the present methods, it may be of interest to note that the present paper was completed before I had seen this most suggestive paper of MAXWELL'S.

I have not hesitated to put the symbolic vector, ∇, among the intensities since it obeys all the laws thereof. The *definitions* of the connection between σ, σ', and σ'', and between τ, τ', and τ'', may be taken as the first and third of equations (5) and (6) respectively. The second and fourth equation of each set are easily deduced from these by observing that $d\rho' = \chi d\rho$, $d\Sigma' = m\chi'^{-1}d\Sigma$ [equations (26) and (37) of former paper], and that both $d\rho$ and $d\Sigma$ are arbitrary vectors.

It should, perhaps, be noticed that these connections between σ, σ', and σ'' and between τ, τ', and τ'', although very useful and intimately connected with the physical nature of the vectors indicated are, after all, only definitions, and thus the phrase "where such and such a symbol is *defined* as a flux" will frequently occur below. This merely means that, having assigned the meaning of one of the three vectors, say τ', by a physical definition, the allied symbols, τ and τ'', are defined by saying that the symbol in question is a flux.

The connection between σ and σ' may be put in words, thus :—*If σ be an intensity, any line integral of σ' referred to the present position of matter is equal to the corresponding line integral of σ referred to the standard position of matter.* Of course, by the word "corresponding" it is implied that the two line integrals are to be taken through the same chains of matter. Similarly as to τ :—*If τ be a flux, any surface integral of τ' referred to the present position of matter is equal to the corresponding surface integral of τ referred to the standard position of matter.*

8. It is convenient to give here the following four simple but useful propositions.

Prop. I. If σ_a, σ_b be two intensities, $V\sigma_a\sigma_b$ is a flux.—By this is meant that $V\sigma_a'\sigma_b'$ bears the same relation to $V\sigma_a\sigma_b$ as does τ' to τ in equations (6). To prove

$$\begin{aligned} Sd\Sigma\sigma_a\sigma_b &= m^{-1}S\chi'd\Sigma'\chi'\sigma_a'\chi'\sigma_b' \text{ [eq. (5) § 7]} \\ &= Sd\Sigma'\sigma_a'\sigma_b' \text{ [TAIT's 'Quaternions,' 3rd ed., § 158, eq. (3)].} \end{aligned}$$

Prop. II. If σ, τ be an intensity and flux respectively, we have $S\sigma\tau d\varsigma = S\sigma'\tau'd\varsigma' = S\sigma''\tau''d\varsigma'$.—For by equations (5) and (6) § 7, $S\sigma'\tau' = m^{-1}S\sigma\tau$, and $S\sigma''\tau'' = S\sigma'\tau'$. As particular cases we have

$$\mathbf{SBH}d\varsigma = \mathbf{SB'H'}d\varsigma', \quad \mathbf{SCA}d\varsigma = \mathbf{SC'A'}d\varsigma', \quad \mathbf{SD\Theta}d\varsigma = \mathbf{SD'\Theta'}d\varsigma' \quad . \quad . \quad . \quad (7).$$

Prop. III. If σ be an intensity $V\nabla\sigma$ is a flux.—By this is meant that $V\nabla'\sigma'$ bears the same relation to $V\nabla\sigma$ as does τ' to τ in equations (6). For any surface

$$\iint Sd\Sigma\nabla\sigma = \int Sd\rho\sigma \text{ [eq. (3) § 5 above]} = \int Sd\rho'\sigma' \text{ [eq. (5)]} = \iint Sd\Sigma'\nabla'\sigma' \text{ [eq. (3)].}$$

Hence, $Sd\Sigma\nabla\sigma = Sd\Sigma'\nabla'\sigma'$, or $V\nabla\sigma$ is a flux.

As particular cases, note that if, as we shall do directly, we assert that

$$4\pi\mathbf{C}' = V\nabla'\mathbf{H}', \quad \mathbf{B} = V\nabla\mathbf{A},$$

and that **B**, **C** are fluxes, and **H**, **A** intensities, it will follow that

$$4\pi\mathbf{C} = \mathrm{V}\nabla\mathbf{H}, \quad \mathbf{B}' = \mathrm{V}\nabla'\mathbf{A}'.$$

Prop. IV. If τ *be a flux* $\mathrm{S}\nabla\tau d\varsigma = \mathrm{S}\nabla'\tau' d\varsigma'$.—Proved by applying eq. (4), as we applied eq. (3) to prove Prop. III. As particular cases notice that

$$\mathrm{S}\nabla\mathbf{D}d\varsigma = \mathrm{S}\nabla'\mathbf{D}'d\varsigma', \quad \mathrm{S}\nabla\mathbf{C}d\varsigma = \mathrm{S}\nabla'\mathbf{C}'d\varsigma' \quad . \quad . \quad . \quad . \quad . \quad . \quad (8).$$

9. Intimately connected with these two classes of vectors are two classes of linear vector functions of a vector.

In the following statements, as indeed throughout the paper, σ will denote an intensity, and τ a flux.

Class I. of Linear Vector Functions of a Vector.

(Examples:—The reciprocal of any function of Class II.; ordinary stress, ϕ, Φ; heat and electric conductivity, γ, R^{-1}; specific inductive capacity, K; magnetic permeability, μ).

Ω being of this class, the three allied symbols, Ω, Ω', Ω'', are connected by the equations

$$\left.\begin{array}{l} \mathrm{S}\sigma_a\Omega\sigma_b d\varsigma = \mathrm{S}\sigma_a'\Omega'\sigma_b' d\varsigma' = \mathrm{S}\sigma_a''\Omega''\sigma_b'' d\varsigma \\ \Omega' = m^{-1}\chi\Omega\chi', \quad \Omega'' = m^{-1}\psi\Omega\psi \end{array}\right\} \quad . \quad . \quad . \quad . \quad . \quad . \quad (9)$$

σ_a and σ_b being any two intensities.

Class II. of Linear Vector Functions of a Vector.

(Example.—The reciprocal of any function of Class I., *e.g.*, electric resistance, R).

Υ being of this class, the three allied symbols Υ, Υ', Υ'' are connected by the equations

$$\left.\begin{array}{l} \mathrm{S}\tau_a\Upsilon\tau_b d\varsigma = \mathrm{S}\tau_a'\Upsilon'\tau_b' d\varsigma' = \mathrm{S}\tau_a''\Upsilon''\tau_b'' d\varsigma' \\ \Upsilon' = m\chi'^{-1}\Upsilon\chi^{-1}, \quad \Upsilon'' = m\psi^{-1}\Upsilon\psi^{-1} \end{array}\right\} \quad . \quad . \quad . \quad . \quad . \quad . \quad (10)$$

τ_a and τ_b being any two fluxes.

Of course, it is understood that Ω' and Υ' are not, as usual, the conjugates of Ω and Υ. Note, that if Ω or Υ is self-conjugate, then Ω' and Ω'' or Υ' and Υ'' are also self-conjugate. The first and second of each of the sets of equations (9) and (10) may be taken as the definitions of Ω', Ω'', Υ', Υ''. The third and fourth equations of each set can easily be proved by equations (5) and (6) to follow.

10. The following easily-proved propositions should be noticed :—

Prop. V. If Ω *be of Class I., then* Ω^{-1} *is of Class II., and if* Υ *be of Class II., then* Υ^{-1} *is of Class I.*

Prop. VI. $\Omega\sigma$ *is a flux, and* $\Upsilon\tau$ *is an intensity.*

Prop. VII. $S\nabla(\Omega\sigma)\,d\varsigma = S\nabla'(\Omega'\sigma')\,d\varsigma'$. (Props. IV. and VI.)

Prop. VIII. $\Omega' d\Sigma'$ *is the same function of* $\Omega d\Sigma$ *as* $d\rho'$ *is of* $d\rho$, *and* $\Upsilon' d\rho'$ *is the same function of* $\Upsilon d\rho$ *as* $d\Sigma'$ *is of* $d\Sigma$ *or* $\Omega' d\Sigma' = \chi\Omega d\Sigma$, $\Upsilon' d\rho' = m\chi'^{-1}\Upsilon d\rho$.

Prop. IX. $\Omega'\Delta' d\varsigma' = \chi\Omega\Delta d\varsigma$. [Prop. VIII. and eq. (4).]

11. Going, now, back to our definition of electric coordinates (§ 4), since for each element they may now be written $S\mathbf{D}d\Sigma_a$, $S\mathbf{D}d\Sigma_b$, $S\mathbf{D}d\Sigma_c$, and since $d\Sigma_a$, &c., are *constants*, we see that the choice of coordinates is equivalent to regarding $\mathbf{D}$ and not $\mathbf{D}'$ as the independent electric variable at any point. Further from eq. (1) § 4, and eq. (6) § 7, we have

$$S\mathbf{C}d\Sigma = dS\mathbf{D}d\Sigma/dt,$$

or, since $d\Sigma$ is an arbitrary *constant* vector,

$$\mathbf{C} = d\mathbf{D}/dt \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (11),$$

which is, of course, inconsistent with the equation $\mathbf{C}' = d\mathbf{D}'/dt$.

C.—*Preliminary Justification of the Foundations of the Present Theory.*

12. I have deliberately led up as quickly as possible to a description of the mathematical machinery to be used subsequently, as it has been necessary to notice incidentally some of the essential characteristics of the fundamental assumptions and the methods of investigating their consequences advocated in the present paper. As a preliminary justification of these assumptions, I cannot do better than indicate the line of thought which led up to them.

In studying MAXWELL's theory, and seeing how beautifully it was built up step by step from a mass of experimental facts, till the consistent whole stood revealed, it seemed to me that, notwithstanding the general harmony of its different parts, there was just here still something to be desired, some single plan that should govern *the whole*. This statement may not seem justifiable, so I instance two examples of the want of harmony. In one part of his treatise, the kinetic part, he works out the connections between the different parts of his theory by the general methods of Dynamics. But not so in the statical part. It would seem that the statical part of the subject, in such a plan as just mentioned, ought to appear as a particular case of the kinetic, whereas, in MAXWELL's treatise, the statical terms in the equations are merely *added on to those deduced from dynamical reasoning.* The same remark applies to the terms necessary to produce the mechanical effects of magnetism [compare for proof § 603 eq. (11) with the corresponding eq. (the last on p. 239) of § 619], and similar remarks, apply to the treatment in § 630 *et seq.* of the energy of the field. How to bring these various parts of the subject under the dynamical treatment I did not see, strictly on MAXWELL's own lines. Again, in considering the general equations of the electromagnetic field (§§ 608, 609) he speaks of a generalised force **E**. This generalised force, quite contrary apparently to dynamical analogies, has, not a single definite effect, partly kinetic and partly static, but two independent effects, one static and the other kinetic. On trying to trace out the reason of this, I could not arrive at any certain result strictly on MAXWELL's own lines. It *seemed* to me as if the double effect of **E** was simply assumed. [If it *merely* were analogous to an ordinary dynamical reaction, then it could not be associated with such external forces as result from electrolysis.]

13. Whether these and many other similar questions which occurred, some of which will appear below, can, strictly speaking, be denominated difficulties, is of no consequence. Suffice it that they led to the following considerations. MAXWELL has built up a theory whose axioms can* be put down in a definite form. Cannot, then, all his results (electrostatic, electrodynamic, magnetic, and electromagnetic) be developed as consequences of these axioms in one application of dynamical reasoning? Cannot we by such a single application obtain all MAXWELL's equations from (A) to (L) in §§ 591 to 614, as well as his stress results contained in other parts of the treatise, and by particular simplifying assumptions, shew that the ordinary electrostatic and magnetic theories are particular consequences of our general results?

This led me to attempt to apply in a perfectly rigorous and general manner the well-known equation

$$\delta\int \mathrm{L}\,dt + \Sigma\int \mathrm{Q}\,\delta q\,dt = 0 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (1)$$

(where L is the Lagrangian function, "modified" if necessary, of any mechanical system of which q is a coordinate, and Q the external force of type q, and where the initial and final positions and times are not subject to variation) to the present case. The way I proposed to apply it was to assume all matter to be in any possible state as to strain and as to electric phenomena, then to vary all the geometrical coordinates by simply giving to each element of matter an arbitrary displacement, and also to vary all the electric coordinates, and trace the mathematical consequences. [Note that on MAXWELL's theory (at least as I understand it) these two variations are all that can be made, a variation in the magnetism being determined by the above variations.]

14. And it was here at the outset that the greatest difficulty of any met with in the investigation occurred. Consider a particular consequence of assuming that the electric coordinates are the three components of **D**′ for every point of space. If by

* It would be more correct to say "*some* of whose axioms." I wish to imply that I thought it advisable to fill in the remainder tentatively and seek the result.

the variation of geometrical coordinates an element ds' of surface where there is finite surface density of electricity be moved from P to P′, then in general the element of matter will *by the variation of the geometrical coordinates only* be entirely deprived of its charge, for this charge will be left behind at P. This result is, to say the least, an unfortunate one, and to be avoided, if by legitimate means it is possible. Still more disastrous results are arrived at if we assume that the components of $\mathbf{D}'$ for every element of *matter* are the electric coordinates, for then the charge in the whole of space is varied by a mere variation of the geometrical coordinates.

The legitimate way out of the difficulty seemed to be to assert that these electric coordinates, though theoretically permissible, were very unsuitable. To find suitable ones it was natural to use the principle that *the electric coordinates must be such that the variation in the geometrical coordinates does not alter the charge of any portion of matter.* This is, of course, ensured by assuming that $S\mathbf{D}'d\Sigma'$ is unaltered by variation of the geometrical coordinates, and from this it is but a step to the assertion that $S\mathbf{D}'d\Sigma'$ is itself a suitable electric coordinate.

Intimately connected with this question of the independent variation of geometrical and electrical coordinates is that of the correct expression for an electric current in (say) an arbitrarily moving fluid. It is not necessary to present all the reasons that occurred to me for the form already described (§ 4) as these are sufficiently indicated in the above considerations of variation of coordinates.

D. *An Analogy.*

15. The resemblances and differences between the present fundamental assumptions and what I take to be MAXWELL'S, are, perhaps, more clearly brought out by analogy.

I will first describe what I understand to be the analogy which MAXWELL allows himself throughout his theory, in order more closely to realise the interdependence of the various physical quantities considered, and as an aid to memory. The analogy contemplates the whole of space as being filled with an incompressible liquid. In dielectrics the liquid is, as it were, held in elastic meshes, in the form of closed cells, so that if it be displaced it tends to return to its original position. In the ideal conductor there are no such meshes, or rather there are meshes which do not form closed cells, so that the liquid can move through them, but is resisted while in motion. An actual body which admits some conduction, but behaves also like a dielectric will be typified by meshes which allow a slow leakage of the liquid. Now suppose into any space we introduce from some external source more liquid. This foreign liquid will be what is called the electric charge of that space, and it may be measured (since the liquid is incompressible) by the surface integral over the boundary of the space considered of the displacement of the original liquid outwards. Thus, "electric displacement" is represented in the analogy by a flux of the liquid.

The "conduction" current is measured by the current of *foreign* liquid, and the "displacement" current (indicated in the present paper by the term "dielectric" current) by that of the original liquid. In a simple conductor there is nothing to distinguish foreign from original liquid, and the conduction current in this case is represented by the whole liquid current.

A similar but not identical analogy will hold in the theory now advocated. For fixed matter the whole of the foregoing would be true, but not for moving matter. The liquid in the present analogy must not be incompressible, but must have a property in connection with *matter* which corresponds to the property of an incompressible liquid with reference to *space*. An incompressible liquid is one of which only one definite quantity can occupy an assigned space. In the present analogy we must say, instead, that the liquid is contained by matter, and that a given portion of matter always contains the same quantity of liquid. If by any means we pump foreign liquid into this portion, then an equal quantity of liquid must pass out of the boundary of that portion of matter into neighbouring matter, and thus in the present analogy as in the former, electric displacement will have for analogue the flux of the liquid, but not as in that case, across a surface fixed in space, but across a surface fixed relatively to matter.

Similar remarks apply to currents.

E. *Plan of the Paper.*

16. It will conduce to clearness to give some account here of the objects and aims of what is to follow. The part of the paper succeeding this introduction is in three main divisions: *The groundwork of the theory; The establishment of general results,* and *The detailed examination of these results.*

The groundwork of the theory, though not the longest of these, calls for most attention here. It is divided into two parts, *Fundamental assumptions* and *Preliminary dynamical and thermodynamical considerations.* I do not propose to give here a résumé of the different parts, but to call attention to certain prominent features.

The two most important of the fundamental assumptions are, perhaps, first, that in all cases $4\pi\mathbf{C} = \mathrm{V}\nabla\mathbf{H}$, which I take to be one of the most characteristic features, if not the most characteristic, of MAXWELL's theory, and secondly, that the modified Lagrangian function per unit volume, though, of course, it contains **H**, does not contain any term involving magnetic moment per unit volume or magnetic induction. Neither of these assumptions seems to be at variance with MAXWELL's, and, as hinted, the first is taken up mainly because it is a fundamental feature in his theory. From the first it follows that **C** must obey the laws of incompressibility, and this naturally leads to the assumption that **D** also invariably obeys those laws. The second leads to very important consequences, which, I believe, have not before been traced, and which I wish to call attention to here. Though not put quite in this form below they amount to this, that ${}_{\mathrm{H}}\nabla l$, where l is the modified Lagrangian function per unit volume of the standard position of matter, obeys the laws of incompressibility that round every circuit there is an electromotive force equal to the rate of decrease of the surface integral of $4\pi_{\mathrm{H}}\nabla l$ through the circuit, and that ${}_{\mathrm{H}}\nabla l - \mathbf{H}/4\pi$ appears in subsequent equations in such a manner as to compel us to identify it with the magnetic moment per unit volume.* It is clear, then, that $4\pi_{\mathrm{H}}\nabla l$ is, according to the present theory, the magnetic induction. As the theory is developed below it is convenient to define **B** as equal to $4\pi_{\mathrm{H}}\nabla l$ and call **B** the magnetic induction, leaving the justification till we examine the detailed consequences of the theory. It is well to insist on this result here, as it does not appear obvious in the work below, but only comes out when a general review of a great part of the paper is made. To put the matter in the form of a proposition :—

If the two fundamental assumptions are made—(1) *that* $4\pi\mathbf{C} = \mathrm{V}\nabla\mathbf{H}$, *and* (2), *that* l, *the Lagrangian function per unit volume, can be expressed in terms involving* **H** *but independent of magnetic induction and of magnetic moment per unit volume, then the magnetic induction must be* $= 4\pi_{\mathrm{H}}\nabla l$.

17. The other most important features of the fundamental assumptions are first those already described with reference to the electric coordinates, and the expression for the current in terms of the displacement; and secondly the manner in which are treated the two currents, conduction and dielectric (the latter being inappropriately, on the present theory, denominated the "displacement current"). If there are (and physicists seem agreed on the point) two independent currents whose *sum* appears in the equation $4\pi\mathbf{C} = \mathrm{V}\nabla\mathbf{H}$, and whose *sum* obeys the laws of incompressibility, it seems to me of the nature of a truism that there must be also two independent electric displacements whose *sum* obeys the laws of incompressibility. I therefore, from the very beginning, recognise two displacements, **d** and **k**, which I call, for want of better names, the dielectric and conduction displacements.† This naturally leads to the contemplation of two independent kinds of electromotive force. This last, however, is subsequently satisfactorily disposed of.

18. Before leaving the fundamental assumptions, let me remark that though in some important respects the present theory may seem to differ from MAXWELL'S, it will be found, I think, that just where the difference seems to be most marked, is MAXWELL'S theory most vague. All the differences, if they really be such, have been forced on me unwillingly in the attempt to put into definite form what I take to be the essence

* Strictly speaking, the last clause should be modified by the condition "if the present position be taken as the standard position." This, however, is only an accident of the particular form of enunciation, which, at the present stage, is unavoidable.

† Perhaps it would be better to call them the *elastic* and *frictional* displacements or the *reversible* and *irreversible* displacements. I wish to leave this point open for those better qualified to decide. Of the three sets of terms suggested above, the last seems to be the best. The only reason for adopting in the present paper the names given in the text is to imply the origin of the assumption that there are *two* such displacements. Of course, if we call the two displacements reversible and irreversible, we must also call the corresponding currents reversible and irreversible.

of MAXWELL'S theory. At any rate the results, though not in every respect identical with MAXWELL'S, are yet so nearly identical that the true matter for surprise is that they differ so little, and in such unimportant ways, from his.

It must be added, to prevent misconception of my own views, that I by no means consider proven what I regard as the key to MAXWELL'S theory, and what I have strictly adhered to in this paper, the assumption that under all circumstances $4\pi\mathbf{C} = \mathrm{V}\nabla\mathbf{H}$. My position rather is, that while this assumption may or may not be true, it is desirable to investigate as generally as possible what must be true, and what cannot be true if the assumption is made. In other words, I do not think that MAXWELL'S theory has yet had a fair trial, even at the hands of mathematicians, and the present paper is an attempt to provide more ways and means than hitherto have been available for such a trial. The methods adopted are equally applicable to other sets of fundamental assumptions.

19. Turning to the second part of the groundwork, the *preliminary dynamical and thermodynamical considerations*, it is necessary to remark that these considerations though not limited to an electric field, seemed absolutely necessary in order thoroughly to investigate the consequences of the assumptions. With regard to the first two sections of this part of the paper on *the modified kinetic energy and the free energy*, and on the *entropy* there is nothing which is likely to be questioned. In the third section on *frictional forces, conduction of heat and dissipation of energy*, I enunciate a principle which opens the way for much criticism. I would beg any readers to whom the form of enunciation is repugnant, to suspend their judgment as to the validity of the principle, not only until the first justification of it, but until they have seen it in action as it were, later in the paper. What was wanted was to bring this group of phenomena, which are undoubtedly closely connected, under the same sort of treatment as is accorded to the reversible phenomena of a system by means of its Lagrangian function, and the (dependent) entropy.

20. The way being thus paved, in the next principal division of the paper are deduced the general results of the theory, the most important of which are the equations of motion. These are considerably more general than the ordinary equations of the field, and thus we are led to the last division of the paper, the detailed examination of these results. The chief sub-divisions of this part are the comparison with MAXWELL'S results, a discussion from the point of view of the present theory of thermoelectric, thermomagnetic, and the HALL effects, and the transference of intrinsic energy through the field.

In comparing with MAXWELL'S results, wherever there is agreement, it is considered unnecessary to investigate further the detailed consequences. Where there is disagreement the physical consequences are traced with more detail, and in no case can it, I think, be said that the results of this part of the paper are condemnatory of the present theory. In this place, too, the bearing of the present theory on the question of convection currents is discussed.

Perhaps a clearer insight into the true bearings of the present theory is obtained by the attempt below to explain thermoelectric, thermomagnetic, and the HALL effects than by any other part of the paper. Especially clearly do some of the restrictions imposed by the condition $4\pi\mathbf{C} = \mathrm{V}\nabla\mathbf{H}$ come out.

21. In the last sub-division it will be found that I disagree entirely with Professor POYNTING'S interpretation of his own results, and show how quite a different and, I think, simpler flux of energy may be made to account for the changes of intrinsic energy in different parts of the field. In particular, this interpretation would restore credence in what Professor POYNTING considers he has shown to be a false view, viz., that among other aspects of a current of electricity it may be looked upon as something conveying energy along the conductor. This part of the subject, although deduced from the present theory, is shown to be true on Professor POYNTING'S own premisses.

22. It is well here to call attention to what might prove confusing otherwise. In what follows $\mathbf{E}$, $\mathbf{e}$, $\mathbf{F}$, Φ, and some allied symbols, stand for certain external forces. But there are three different meanings given in different parts of the paper to these symbols. They are originally defined as the whole external forces of the different types. But in treating of frictional forces, &c. (§§ 35 to 42) it is convenient to regard them as meaning only those parts of the forces which are *due to* friction and the like. Again from § 50 onwards it is convenient to regard them as meaning only those parts of the forces which are *independent of* friction and the like. This inconvenience is incurred to avoid the greater evil of a large additional array of symbols.

With this exception,* and one or two other trifling ones, which are noticed in their places, nowhere has the meaning of a symbol been changed throughout the paper.

II. GROUNDWORK OF THEORY.

A. *Fundamental Assumptions.*

23. We assume that the Lagrangian function, L, of all matter in space can be expressed in the form

$$\mathrm{L} = \iiint l d\varsigma + \iint l_s ds \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (1),$$

where l, l_s are functions of certain independent variables which determine the state of the body at the point. The volume integral extends throughout space, and the surface integral over certain specified surfaces. The entropy F of all matter in space will be assumed to be of the form

* Since completing the paper I have discovered a notable exception which is *not* otherwise noted than in this footnote. It does not seem likely to lead to confusion; therefore I retain it. Most frequently in the present paper q stands for the typical *scalar* coordinate of a dynamical system, but it is not infrequently used, as in the former paper, for the *quaternion* of the rotation-operator $q(\ )q^{-1}$.

$$F = \iiint f d\varsigma + \iint f_s ds \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (2),$$

and f and f_s will be determined from the values of l and l_s in a manner that will be described later on. All thermal phenomena not determined by F, and all forces of the nature of friction, will be supposed given by a third function X, given by

$$X = \iiint x d\varsigma + \iint x_s ds \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (3),$$

where x, x_s, unlike f, f_s, do not in any way depend upon L. The way in which these forces and the thermal phenomena depend upon X will be explained later. We shall call X the dissipation function. It is, in fact, a generalisation of Lord RAYLEIGH's dissipation function ('Theory of Sound,' 1st ed., § 81).

24. The absolute temperature of any element of matter will be denoted by θ. The vector Θ (assumed an intensity—§ 7 above) is defined by the equation

$$\Theta = \nabla\theta \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (4).$$

Since both Θ and ∇ are intensities, we have

$$\Theta' = \nabla'\theta \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (5).$$

All electric and magnetic phenomena are supposed ultimately to depend upon the magnitudes and rates of variation of two fluxes (§ 7), **d**, **k**, called respectively the dielectric displacement and the conduction displacement. The whole displacement, **D**, is defined as the sum of these two, so that

$$\mathbf{D} = \mathbf{d} + \mathbf{k} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (6).$$

D must satisfy the two conditions of incompressibility for vectors, *i.e.*,

$$S\nabla\mathbf{D} = 0, \quad [Sd\Sigma\mathbf{D}]_{a+b} = 0 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (7),$$

the notation $[\ \ ]_{a+b}$ being as defined on p. 119 of former paper, *i.e.* the suffixes a and b denote the two regions bounded by a surface of discontinuity, and $[\ \ ]_{a+b}$ stands for $[\ \ ]_a + [\ \ ]_b$. Since **D** is a flux, it follows by Prop. IV., § 8, above, that

$$S\nabla'\mathbf{D}' = 0, \quad [Sd\Sigma'\mathbf{D}']_{a+b} = 0 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (8).$$

The dielectric current, **c**, the conduction current, **K**, and the whole current, **C**, all assumed to be fluxes, are given by the equations

$$\mathbf{c} = \dot{\mathbf{d}}, \quad \mathbf{K} = \dot{\mathbf{k}} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (9),$$

$$\mathbf{C} = \dot{\mathbf{D}} = \mathbf{c} + \mathbf{K} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (10).$$

25. The differentiations with regard to time implied by these dots are differentiations for a fixed element of the standard position of matter, *i.e.*, they are differentiations that follow the motion of matter. It is clear then that they are commutative with ∇, but not with ∇'. Hence, from equations (7) and (10),

$$S\nabla\mathbf{C} = 0, \qquad [Sd\Sigma\mathbf{C}]_{a+b} = 0 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (11),$$

and, therefore, [equation (8) § 8 above]

$$S\nabla'\mathbf{C}' = 0, \qquad [Sd\Sigma'\mathbf{C}']_{a+b} = 0 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (12).$$

Since $\mathbf{C}$ satisfies the conditions of incompressibility, its surface integral over any surface only depends on the boundary of the surface, and may be expressed as the line integral of a vector $\mathbf{H}/4\pi$ round it. Thus, by equation (3), § 5,

$$4\pi\mathbf{C} = V\nabla\mathbf{H} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (13).$$

$\mathbf{H}$ is called the magnetic force, and is assumed to be an intensity, so that (Prop. III., § 8)

$$4\pi\mathbf{C}' = V\nabla'\mathbf{H}' \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (14).$$

All the vectors, including $\mathbf{H}$, hitherto mentioned, may be discontinuous. But they are assumed to be finite, so that $\iiint\mathbf{C}d\varsigma = 0$ for any infinitely small volume. Suppose this volume is a disc enclosing a part of a surface of discontinuity in $\mathbf{H}$. Then we have

$$0 = \iiint V\nabla\mathbf{H}d\varsigma = \iint Vd\Sigma\mathbf{H}$$

by equation (4), §5 above. Hence

$$[Vd\Sigma\mathbf{H}]_{a+b} = 0 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (15),$$

so that the discontinuity in $\mathbf{H}$ is entirely normal to the surface. Similarly

$$[Vd\Sigma'\mathbf{H}']_{a+b} = 0 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (16).$$

26. From what has been said it follows that if $\mathbf{d}$, $\mathbf{k}$ and their rates of variation are given for every point of space, $\mathbf{H}$ is not yet completely determined. It is, however, so determined by one more condition which is proved in § 48 below, and which is given here as we shall want to use it before proving it. $\mathbf{H}$ is one of the independent variables of which l is supposed an explicit function. The condition mentioned is that ${}_{\mathbf{H}}\nabla l$ satisfies the conditions of incompressibility. In other words, putting

$$4\pi_{\text{H}}\nabla l = \mathbf{B} \quad \ldots \quad (17),$$

$$S\nabla\mathbf{B} = 0, \qquad [Sd\Sigma\mathbf{B}]_{a+b} = 0 \quad \ldots \quad (18),$$

and **B** is called the magnetic induction.

These are proved by previously proving that

$$\mathbf{B} = V\nabla\mathbf{A} \quad \ldots \quad (19),$$

where **A** is a vector which satisfies the condition

$$[Vd\Sigma\mathbf{A}]_{a+b} = 0 \quad \ldots \quad (20).$$

A is assumed to be an intensity, and **B** a flux, so that (§ 8 above),

$$S\nabla'\mathbf{B}' = 0 \quad , \quad [Sd\Sigma'\mathbf{B}']_{a+b} = 0 \quad \ldots \quad (21),$$

$$\mathbf{B}' = V\nabla'\mathbf{A}' \quad , \quad [Vd\Sigma'\mathbf{A}']_{a+b} = 0 \quad \ldots \quad (22).$$

This relation between **B** and **H** is not the usually accepted one, but it is certainly true on the present theory. It will appear later on that the value thus arrived at of **B**$'$, the magnetic induction at the point ρ', is independent of the particular position which is chosen as a standard of reference.

In the present theory **I**—assumed a flux—called the magnetic moment per unit volume is defined by the equation

$$\mathbf{B}' - \mathbf{H}' = 4\pi\mathbf{I}' \quad \ldots \quad (23),$$

from which it does not follow that $\mathbf{B} - \mathbf{H} = 4\pi\mathbf{I}$, since **B** and **I** are fluxes and **H** is an intensity. It does follow, however, that

$$\mathbf{B}'' - \mathbf{H}'' = 4\pi\mathbf{I}'' \quad \ldots \quad (24).$$

27. The equations of last article, it will be observed, do not represent fundamental assumptions. They are given here merely to indicate how the familiar symbols involved appear in the present theory. We now return to the fundamental assumptions.

The independent* variables, of which l is supposed a given explicit function, are

$$\theta, \Theta; \ \rho', \dot{\rho}', \Psi; \ \mathrm{d}, \mathbf{D}, \mathbf{C}, \mathbf{H} \quad \ldots \quad (25);$$

x is supposed a given explicit function of

$$\theta, \Theta; \ \Psi, \dot{\Psi}; \ \mathbf{K}, \mathbf{H} \quad \ldots \quad (26).$$

* See § 31 below.

Nothing is here said about l_s and x_s, as it has been thought advisable to see how much can be explained without their aid. When we come to consider electrostatic contact potential difference—which for brevity we will in the future call contact-force —it will be found necessary to suppose l_s not zero. For ordinary friction, also, x_s must not be zero. The above assumptions will enable us to take account of (1) all MAXWELL'S results, or results corresponding thereto; (2) the stresses, &c., resulting from variation of specific inductive capacity and magnetic permeability with strain and temperature; (3) thermoelectric, thermomagnetic, and the HALL effects; (4) many purely mechanical results whose details will be reserved for future treatment. They do not enable us to take account of (1) sliding friction; (2) electrolysis; (3) hysteresis and similar phenomena; (4) contact-force. All these, however, except (3), can be taken account of by slight additions to our present assumptions, as in the case of (4) will appear later.

The object of limiting as above the number of the independent variables entering into l and x is to free the mind from unnecessary vagueness. Moreover, the above assumptions are in one sense necessarily simpler than those made by Professor J. J. THOMSON ('Applications of Dynamics to Physics and Chemistry,' 1st ed., chap. vii.) to explain thermoelectric and thermomagnetic effects, in that the only quantity whose space-variations appear above in l or x is θ, a statement not true of Professor THOMSON'S assumptions. With regard to the forms of l and x as functions of their independent variables, it is simplest at present to make no restrictions.

28. I am a little doubtful whether writers on the present subject recognize two semi-independent electric displacements at every point, but, as already remarked, it seems to me to follow, as a matter of course, from the assumption of two independent currents. The independent variables which have [§ 27 (25)] above been chosen to take account of these are **d** and **D**, though, of course, **d** and **k** or **k** and **D** might have been chosen instead. MAXWELL generally, but not quite without exception, seems to use the symbol **D** for what I have called **d**. I thought, however, that I should be following the usual custom of subsequent writers by using **D** for the whole displacement.

If there be two independent electrical displacements, it would seem as though we must assume, at any rate provisionally, the existence of two independent external electromotive forces of type **D** and **d**. These we shall denote by $-\mathbf{E}$ and $-\mathbf{e}$ respectively. This, of course, means that the work done by the said external forces at the element $d\varsigma$, while **D**, **d** change to $\mathbf{D} + d\mathbf{D}$ and $\mathbf{d} + d\mathbf{d}$ respectively, is $(S\mathbf{E}\, d\mathbf{D} + S\mathbf{e}\, d\mathbf{d})\, d\varsigma$. We shall also assume external surface forces of these types $-\mathbf{E}_s$, $-\mathbf{e}_s$, external ordinary forces **F** and $\mathbf{F}_s$ per unit volume and surface of the present position of matter,* and an external stress Φ, Φ being a self-conjugate linear vector function of Class I of § 9 above. This last statement means that the real

* In the former paper $\mathfrak{F}$ meant the force per unit volume of the *standard* position of matter. The change has been made, since the equations of this paper are thereby simplified.

stress-function is not Φ, but Φ', *i.e.*, that the force exerted on a region at the element $d\Sigma'$ of its boundary is $\Phi' \, d\Sigma' = m^{-1}\chi\Phi\chi' \, d\Sigma'$.

29. The meaning of "external" may be defined as "not included in the form of L." Thus, the external forces include (1) all frictional forces given by X; (2) forces that, though not now included in the form of L, can be so included by generalising the meaning of l and l_s, so as to explain electrolysis, contact-force, and chemical phenomena; (3) forces that, through present ignorance we cannot include in X or L, though they should be so included. Thus, for instance, the external stress Φ may be supposed to be due entirely to viscosity and elastic fatigue, and the first of these will be accounted for by X.

30. $\delta\mathbf{D}'$ is due partly to variation of strain and partly to $\delta\mathbf{D}$; let $\delta'\mathbf{D}'$ be the latter part. $\mathbf{E}$ and $\mathbf{e}$ are assumed to be intensities. Hence (§ 8, Prop. II.)

$$(\mathrm{S}\mathbf{E}\,\delta\mathbf{D} + \mathrm{S}\mathbf{e}\,\delta\mathbf{d})\,d\varsigma = (\mathrm{S}\mathbf{E}'\,\delta'\mathbf{D}' + \mathrm{S}\mathbf{e}'\,\delta'\mathbf{d}')\,d\varsigma' \quad . \quad . \quad . \quad . \quad . \quad . \quad (27).$$

A similar theorem is supposed to hold with regard to $\mathbf{E}_s$, $\mathbf{e}_s$, viz.:

$$(\mathrm{S}\mathbf{E}_s\,\delta\mathbf{D} + \mathrm{S}\mathbf{e}_s\,\delta\mathbf{d})\,ds = (\mathrm{S}\mathbf{E}'_s\,\delta'\mathbf{D}' + \mathrm{S}\mathbf{e}'_s\,\delta'\mathbf{d}')\,ds' \quad . \quad . \quad . \quad (28),$$

from which, since [§ 7, eq. (6)] $\delta'\mathbf{D}' = m^{-1}\chi\,\delta\mathbf{D}$, and

$$ds'/ds = \mathrm{T}\,d\Sigma'/\mathrm{T}\,d\Sigma = m\mathrm{T}\chi'^{-1}\mathrm{U}\nu = m\mathrm{T}^{-1}\chi'\mathrm{U}\nu'$$

$$\mathbf{E}'_s = \chi'^{-1}\mathbf{E}_s/\mathrm{T}\chi'^{-1}\mathrm{U}\nu, \quad \mathbf{E}_s = \chi'\mathbf{E}'_s/\mathrm{T}\chi'\mathrm{U}\nu' \quad . \quad . \quad . \quad . \quad . \quad (29),$$

and similarly for $\mathbf{e}_s$, $\mathbf{e}'_s$.

31. We must distinguish carefully between the independent variables of an element of matter which are given in the two lists (25) and (26) of § 27 and the independent variables of the system in general. These last consist only of

$$\theta, \ \rho', \ \mathbf{d}, \ \mathbf{D} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (30),$$

for every element of matter, for when these last and their time-rates of variation are assigned for all space, all the other quantities are determined. [It is not quite correct to talk of $\mathbf{D}$ as an *independent* variable on account of the equations of condition (7) of § 24.]

To enable us to develop the consequences of these fundamental assumptions, a digression on dynamics and thermodynamics must be made.

B. *Preliminary Dynamical and Thermodynamical Considerations.*

Ba. *The "Modified Kinetic" Energy and the "Free" Energy.*

32. It is not to be supposed that the coordinates we have assumed are sufficient to fix the position of all matter in space. The mathematical machinery we use cannot be supposed sufficiently fine to trace the motion of molecules. Such coordinates as would be required for that purpose are "ignored." Now (LARMOR, 'Proc. London Math. Soc.,' vol. 15, 1884, p. 173) in order that the principle expressed in eq. (1) of § 13, above, may be true under these circumstances, L must be, not the true Lagrangian function, but what ROUTH ('Elem. Rig. Dyn.,' 4th ed., § 420) has called a modified Lagrangian function. And that our principle may be true the particular type of modification is assigned, *i.e.*, the ignored coordinates are those whose momenta appear explicitly. And a further restriction is necessary (LARMOR, as above), viz., that the ignored coordinates must *only* appear through their momenta. That is, the ignored coordinates must be what Professor J. J. THOMSON ('Applications,' 1st ed., § 7) has called kinosthenic or speed coordinates. This last restriction, however, is not absolutely necessary if we take L to be the average value of the modified Lagrangian function for a small time, sufficiently large to allow the molecules to go through all their types of motion many times.

33. Whether these restrictions be imposed or not we have the following relation :—

$$\Lambda = \Sigma \dot{q}\, \partial L/\partial \dot{q} - L \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (1),$$

where Λ is the whole energy of the motion due to a modified function L, and q is a coordinate whose velocity appears explicity. (Notice that *if* Λ were supposed expressed, not as a function of the $\dot{q}$'s, but as a function of the $\partial L/\partial \dot{q}$'s, it would be the reciprocal function of L with regard to the $\dot{q}$'s [ROUTH's 'Elem. Rig. Dyn.,' 4th ed., § 410.] It is not this reciprocal function only, because, for our purposes, it is more convenient to assume it an explicit function of the same quantities as L). To prove this, let* ϕ, Φ be a coordinate, whose momentum appears, and its momentum respectively, and let L_0 be the Lagrangian function of which L is the modified form. Thus

$$\Sigma \dot{q}\, \partial L/\partial \dot{q} - L = \Sigma \dot{q}\, \partial L_0/\partial \dot{q} - (L_0 - \Sigma \dot{\phi} \Phi).$$

[ROUTH's 'El. Rig. Dyn.,' 4th ed., §§ 410, 420].

$$= \Sigma\, (\dot{q}\, \partial L_0/\partial \dot{q} + \dot{\phi}\, \partial L_0/\partial \dot{\phi}) - L_0. \quad [\textit{ibid.}]$$

$$= 2\mathfrak{T} - (\mathfrak{T} - \mathfrak{V}). \quad [\mathfrak{T} = \text{kinetic energy}, \mathfrak{V} = \text{potential energy.}]$$

$$= \mathfrak{T} + \mathfrak{V} = \Lambda.$$

* There is no danger of confusion of these meanings of ϕ, Φ with the stress meanings these symbols bear through the rest of this paper.

Equation (1) can be put in a more convenient form for our purpose. We have

$$\delta \mathrm{L} = \frac{d}{dt} \Sigma \frac{\partial \mathrm{L}}{\partial \dot{q}} \delta q + \Sigma \left(\frac{\partial \mathrm{L}}{\partial q} - \frac{d}{dt} \frac{\partial \mathrm{L}}{\partial \dot{q}} \right) \delta q + \Sigma \left(\frac{\partial \mathrm{L}}{\partial \phi} \delta \phi + \frac{\partial \mathrm{L}}{d \Phi} \delta \Phi . \right)$$

With the restrictions just mentioned, we have $\partial \mathrm{L}/\partial \phi = 0$ and $\delta \Phi = 0$. Hence

$$\left. \begin{aligned} \delta \mathrm{L} &= \frac{d}{dt} \Sigma \frac{\partial \mathrm{L}}{\partial \dot{q}} \delta q + \Sigma \left(\frac{\partial \mathrm{L}}{\partial q} - \frac{d}{dt} \frac{\partial \mathrm{L}}{d \dot{q}} \right) \delta q \\ &= 2 \frac{d}{dt} \mathfrak{T} (\delta q_a, \delta q_b, \ldots) + \Sigma \left(\frac{\partial \mathrm{L}}{\partial q} - \frac{d}{dt} \frac{\partial \mathrm{L}}{\partial \dot{q}} \right) \delta q \end{aligned} \right\} \quad . \quad . \quad . \quad . \quad (2),$$

where $2\mathfrak{T}$ $(\delta q_a, \delta q_b \ldots)$ is to be defined as *the function which appears under the operator* d/dt *when* $\delta \mathrm{L}$ *is expressed as the sum of two quantities, one of which is a linear function of the variations of the retained coordinates, and the other is the rate of variation of a similar function.* We now have

$$\Lambda = 2\mathfrak{T} (\dot{q}_a, \dot{q}_b, . \quad) - \mathrm{L} \quad . \quad . \quad \quad . \quad . \quad . \quad . \quad . \quad . \quad (3).$$

We shall show how, for our particular system, $\mathfrak{T} (\dot{q}_a \ldots)$ can be expressed in the form

$$\mathfrak{T} (q_a, q_b, \ldots) = \iiint \mathfrak{t} \, d\varsigma + \iint \mathfrak{t} \, ds \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (4),$$

where $\mathfrak{t}$, $\mathfrak{t}_s$ *are functions of the same independent variables as* l, l_s.

It is convenient to call $\mathfrak{T} (\dot{q}_a \ldots)$ the whole modified kinetic energy, and $\mathfrak{t}$, $\mathfrak{t}_s$ the modified kinetic energies per unit volume and surface respectively. And, similarly putting

$$\lambda = 2\mathfrak{t} - l, \quad \lambda_s = 2\mathfrak{t}_s - l_s \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (5),$$

we shall call λ and λ_s the *free* energy* per unit volume and surface respectively. We thus have

$$\Lambda = \iiint \lambda d\varsigma + \iint \lambda_s ds \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (6).$$

We shall then *assume* that the energy in any *finite region* is the integral on the right of this equation for that region. The surface integral in this case, of course, only applies to surfaces of discontinuity (as to physical quantities) in this region, and not to the true boundary of the region.

* This term is adopted as a translation of HELMHOLTZ's '*freie Energie*' ('Wiss. Abh.,' II., 959). It is not, of course, the same as the intrinsic energy which we are about to determine by a method analogous to HELMHOLTZ's.

Bb. *The Entropy.*

34. If Q be the external force of type q, we have

$$d\Lambda = \Sigma\, Q dq,$$

where dq is the actual increment in q during the element of time. This can, of course, be proved directly from eq. (2). It must now be remembered that all the above variations are only true if we suppose the temperature of every element of matter kept constant. In other words, the last equation must, when we do not make this restriction, be replaced by

$$d\Lambda - d_\theta \Lambda = \Sigma\, Q dq \quad \ldots \ldots \ldots \ldots \quad (7),$$

where $d_\theta\Lambda$ stands for that part of the increment in Λ which is due to increment in temperature in all elements of matter during the element of time. Let now E be the intrinsic energy (including under this term the ordinary kinetic energy of matter as well as all other forms of energy) of all the matter in space. Thus by the fundamental property of entropy (TAIT's 'Heat,' §§ 377, 378),

$$dE = \Sigma\, Q dq + \iiint \theta df d\varsigma + \iint \theta df_s ds \quad \ldots \ldots \ldots \quad (8),$$

whence, from the last equation,

$$d\left(-E + \Lambda + \iiint \theta f d\varsigma + \iint \theta f_s ds\right) = d_\theta\Lambda + \iiint f d\theta d\varsigma + \iint f_s d\theta ds \quad . \quad (9).$$

λ_s will be seen later on to be independent of Θ. Hence

$$\begin{aligned} d_\theta \Lambda &= \iiint \left(\frac{\partial\lambda}{\partial\theta} d\theta - Sd\Theta_\circledcirc \nabla\lambda\right) d\varsigma + \iint \frac{\partial\lambda_s}{\partial\theta} d\theta ds \\ &= \iiint \left(\frac{\partial\lambda}{\partial\theta} + S\nabla_\circledcirc\nabla\lambda\right) d\theta d\varsigma + \iint \left(\frac{\partial\lambda_s}{\partial\theta} - [SU\nu_\circledcirc\nabla\lambda]_{a+b}\right) d\theta ds, \end{aligned}$$

[by putting $d\Theta = \nabla d\theta$, and applying § 5, eq. (4)]. Thus eq. (9) becomes

$$\left.\begin{aligned} & d\left(-E + \Lambda + \iiint \theta f d\varsigma + \iint \theta f_s ds\right) \\ & \quad = \iiint \left(f + \frac{\partial\lambda}{\partial\theta} + S\nabla_\circledcirc\nabla\lambda\right) d\theta d\varsigma + \iint \left(f_s + \frac{\partial\lambda_s}{\partial\theta} - [SU\nu_\circledcirc\nabla\lambda]_{a+b}\right) d\theta ds \end{aligned}\right\} (10).$$

Since the left of this equation is a perfect differential, so is the right. Hence we see that

$$f + \partial\lambda/\partial\theta + S\nabla_\circledcirc\nabla\lambda \quad \text{and} \quad f_s + \partial\lambda_s/\partial\theta - [SU\nu_\circledcirc\nabla\lambda]_{a+b}$$

must be functions of θ only. And further, by including these functions in $-\partial l/\partial\theta$ and $-\partial l_s/\partial\theta$ respectively—a proceeding that will not affect the equations of motion deduced from the form of L—we see that each of these quantities may be put equal to zero. With this extended meaning of L, then

$$f = -\partial\lambda/\partial\theta - S\nabla_{\Theta}\nabla\lambda \quad \text{. . . . . (11)},$$

$$f_s = -\partial\lambda_s/\partial\theta + [SU\nu_{\Theta}\nabla\lambda]_{a+b} \quad \text{. . . . . (12)}.$$

We now see also from eq. (10) that

$$E = \Lambda + \iiint \theta f d\varsigma + \iint \theta f_s ds \quad \text{. . . . . (13)},$$

or

$$E = \iiint e d\varsigma + \iint e_s ds \quad \text{. . . . . (14)},$$

where

$$e = \lambda + \theta f = \lambda - \theta\partial\lambda/\partial\theta - \theta S\nabla_{\Theta}\nabla\lambda \quad \text{. . . . . (15)},$$

$$e_s = \lambda_s + \theta f_s = \lambda_s - \theta\partial\lambda_s/\partial\theta + \theta[SU\nu_{\Theta}\nabla\lambda]_{a+b} \quad \text{. . . . (16)},$$

so that e, e_s may be called the intrinsic energy per unit volume and surface respectively.

Bc. *Frictional Forces, Conduction of Heat, and Dissipation of Energy.*

35. It has already [§ 27 (26)] been mentioned that x is a function of

$$\theta, \Theta;\ \Psi, \dot{\Psi};\ \mathbf{K}, \mathbf{H} \quad \text{. . . . . (17)}.$$

Of these Ψ and **K** are of the nature of velocities, and from the equation $4\pi\mathbf{G} = V\nabla\mathbf{H}$ the same may be said of **H**. Let us, then, briefly speak of them as "the velocities" involved in x. Similarly in the general theory where x_s is not assumed zero, it also will involve certain variables for like reasons called velocities. Let ξ, ξ_s be the functions which are reciprocal (ROUTH's 'El. Rig. Dyn.,' 4th ed., § 410) with regard to Θ and the velocities, to the functions x and x_s. Thus

$$x + \xi = -S\Theta_{\Theta}\nabla x - S\mathbf{K}_{\mathbf{K}}\nabla x - S\mathbf{H}_{\mathbf{H}}\nabla x - S\dot{\Psi}\zeta_{\dot{\Psi}}\mathrm{C\!\!\!\!\,I} x\zeta^{*} \ldots \quad \text{. . (18)},$$

* This seems a good opportunity to place on record a suggestion. There are some obvious objections to the method used in the present and former papers of indicating the independent variable of differentiation of a ∇ or $\mathrm{C\!\!\!\!\,I}$ by an affix. It is somewhat hard to distinguish between **C** and **c** in the

and a similar equation would hold with regard to $x_s + \xi_s$, which, however, requires definite information as to the velocities involved in x_s. ξ is supposed (ROUTH, *ibid.*) expressed as a function, not of the variables (17), but of

$$\theta,\ {}_{\Theta}\nabla x\ ;\ \Psi,\ {}_{\Psi}\mathfrak{C}x\ ;\ {}_{\mathrm{K}}\nabla x,\ {}_{\mathrm{H}}\nabla x \quad \ldots \ldots \quad (19).$$

It is best at first to regard x as a function, not of $\mathbf{K}$, but of $\mathbf{C}$ and $\mathbf{c}$ [§ 24, eq. (10)]. When later we make the assumption that, so far as it depends on these last two, it is a function of their difference ($\mathbf{K}$) only, it will only have to be noticed that

$$_{\mathrm{C}}\nabla x = {}_{\mathrm{K}}\nabla x = -{}_{\mathrm{c}}\nabla x \quad \ldots \ldots \quad (20).$$

Note that this gives

$$\mathrm{S}\mathbf{K}_{\mathrm{K}}\nabla x = \mathrm{S}\mathbf{C}_{\mathrm{C}}\nabla x + \mathrm{S}\mathbf{c}_{\mathrm{c}}\nabla x \quad \ldots \ldots \quad (21),$$

which shows [eq. (18)] that the statement that ξ is the reciprocal of x with regard to Θ and the velocities is still true.

Similarly, if it were assumed, as on a future occasion it will be assumed, that x_s was a function of

$$\theta\ ;\ [\dot{\rho}']_{a-b},\ \Psi_a,\ \Psi_b\ ;\ \mathrm{TSU}\nu\mathbf{K} \quad \ldots \ldots \quad (22),$$

it would be best first to regard it as a function of

$$\theta\ ;\ \dot{\rho}_a',\ \dot{\rho}_b',\ \Psi_a,\ \Psi_b\ ;\ \mathbf{C}_a,\ \mathbf{C}_b,\ \mathbf{c}_a,\ \mathbf{c}_b \quad \ldots \ldots \quad (23),$$

and later make the necessary restrictions.

36. We shall now suppose that the symbols Q, $\mathbf{F}$, $\mathbf{F}_s$, Φ, $\mathbf{E}$, $\mathbf{E}_s$, $\mathbf{e}$, $\mathbf{e}_s$ stand for those parts only of the external forces of the various types, which are owing to friction and the like. To determine their values we shall use the principle*

present paper and Ψ and ψ in the former paper [paragraph following eq. 40)] when used as affixes. There are objections from the printer's and proof-reader's point of view when the affix is anything other than a mere letter. For instance, ${}_{\Psi}\mathfrak{C}$ in the present case, and, still more, ${}_{\dot{\rho}'}\nabla$ in eq. (2), § 44, below, are objectionable on these grounds. Is it not, then, desirable to have, at any rate, an alternative notation? As an alternative to ${}_{\sigma}\nabla$, let me here suggest any one of the following: $\nabla|\sigma|$ $|\nabla\sigma|$ $[\nabla\sigma]$ $\nabla|\sigma$ $\nabla;\sigma$ $\nabla\cdot|\cdot\sigma$ $\nabla\sigma|$ $\nabla\sigma;$. Of these I should personally be inclined to favour $[\nabla\sigma]$ or $\nabla\sigma;$, the latter rather than the former. For instance, in this notation, eq. (18) would become

$$x + \xi = -\mathrm{S}\Theta\nabla\Theta\,;x - \mathrm{S}\mathbf{K}\nabla\mathbf{K}\,;x - \mathrm{S}\mathbf{H}\nabla\mathbf{H}\,;x - \mathrm{S}\dot{\Psi}\zeta\mathfrak{C}\dot{\Psi}\,;x\zeta,$$

which, I think, shows that the notation is sufficiently striking, while it has the advantage of great simplicity.

* I have not been able to reduce this to simpler form or to substitute a simpler principle leading to the same results. I merely wished to make all the phenomena of the kind now being considered depend on some single scalar function X, much as the reversible phenomena depend on the single scalar

$$\iiint\left\{\theta\left(\delta\frac{x}{\theta}+\frac{\partial}{\partial\theta}\frac{\xi}{\theta}\cdot\delta\theta\right)+\dot{f}\delta\theta\right\}d\varsigma$$
$$+\iint\left\{\theta\left(\delta\frac{x_s}{\theta}+\frac{\partial}{\partial\theta}\frac{\xi_s}{\theta}\cdot\delta\theta\right)+\dot{f}_s\,\delta\theta\right\}ds+\Sigma\mathrm{Q}\,\delta\dot{q}=0\quad\ldots\quad(24),$$

where now δ only implies such variations as are the consequences of varying *the velocities of the dynamical system and the temperature*, and where of course $\Sigma\mathrm{Q}\,\delta\dot{q}$ is given in our case by

$$\Sigma\mathrm{Q}\,\delta\dot{q}=\iiint\{-\mathrm{S}\mathbf{F}\,\delta\dot{\rho}'\,d\varsigma'+(\mathrm{S}\mathbf{E}\,\delta\mathbf{C}+\mathrm{S}\mathbf{e}\,\delta\mathbf{c})\,d\varsigma\}$$
$$+\iint\{-\mathrm{S}\mathbf{F}_s\,d\dot{\rho}'\,ds'+(\mathrm{S}\mathbf{E}_s\,\delta\mathbf{C}+\mathrm{S}\mathbf{e}_s\,\delta\mathbf{c})\,ds\}\quad\ldots\quad(25).$$

This last equation would have to be modified if we contemplated finite sliding of one surface over another. In this paper, as already stated, we simplify by supposing this never to take place (except in § 64 below).

Equation (24) is more general than in this paper is required. Throughout this paper x_s, and therefore ξ_s, will be assumed zero.

37. The truth of the principle can be verified (as, admitting the restrictions just mentioned, will be shown directly) by proving that its consequences are in complete harmony with three recognised principles :—(1) that frictional forces can be explained by what Lord RAYLEIGH ('Sound,' 1st ed., vol. I., § 81) calls a dissipation function; (2) that the heat which is created by the destruction of energy in other forms, appears, in the first instance, at the elements of matter where the destruction takes place; (3) the fundamental principle of conduction of heat, that the rate of flow of heat out of any region across the element $d\Sigma'$ of its boundary $=\mathrm{S}\,d\Sigma'\gamma'\Theta'$ where γ' is a self-conjugate linear vector function, which is itself a function of the state of the medium at the point.

38. To show the truth of these statements in the limited circumstances mentioned, viz., when x_s is zero and there is no slipping, notice first what the effects of varying $\mathbf{C}$ and $\mathbf{c}$ only are. A variation in $\mathbf{C}$ will cause a variation in $\mathbf{H}$, since $4\pi\mathbf{C}=\mathrm{V}\nabla\mathbf{H}$ and $[\mathrm{VU}\nu\mathbf{H}]_{a+b}=0$. The device used in the calculus of variations to take account

function L. When there are heat sources not included in our system (L and X) we ought to put $\dot{f}-h$ and $\dot{f}_s-h_s$ instead of $\dot{f}$ and $\dot{f}_s$ in eq. (24), $h\theta$ and $h_s\theta$ being the rate of supply of external heat per unit volume and surface respectively. The form of eq. (24) would perhaps be made more instructive by grouping together the terms

$$\iiint\dot{f}\,\delta\theta\,d\varsigma+\iint\dot{f}_s\,\delta\theta\,ds+\Sigma\mathrm{Q}\,\delta\dot{q}.$$

If $\dot{\mathrm{H}}$ be the rate of "absorption of heat" by a body ($d\varsigma$ or ds) of the system, this expression transforms into $\Sigma\,(\dot{\mathrm{H}}\,\delta\theta/\theta+\mathrm{Q}\delta\dot{q})$.

of such equations of condition is well known. In the present case it takes the following form: to the left of equation (24) add

$$-\iiint \mathrm{S}\mathbf{a}\,(\delta\mathbf{C} - \mathrm{V}\nabla\,\delta\mathbf{H}/4\pi)\,d\varsigma + (4\pi)^{-1}\iint \mathrm{S}\mathbf{a}_s\,d\Sigma\,\delta\mathbf{H}$$

where $\mathbf{a}$, $\mathbf{a}_s$ are vectors; $\delta\,\mathbf{C}$ and $\delta\,\mathbf{H}$ may *then* be regarded as independent. It is to be noted that there is but one $\mathbf{a}_s$ for an element of the bounding surface, *i.e.*, there is not one for each region bounded. In our notation this may be expressed by saying that $[\mathbf{a}_s]_a = [\mathbf{a}_s]_b$.

Now, by eq. (4), § 5, above,

$$\iiint \mathrm{S}\mathbf{a}\nabla\,\delta\mathbf{H}\,d\varsigma + \iint \mathrm{S}\mathbf{a}_s\,d\Sigma\,\delta\mathbf{H} = \iiint \mathrm{S}\,\delta\mathbf{H}\nabla\mathbf{a}\,d\varsigma + \iint \mathrm{S}\,(\mathbf{a}_s + \mathbf{a})\,d\Sigma\,\delta\mathbf{H}.$$

Hence, since the part contributed to the left of eq. (24) by $\delta\mathbf{H}$ is $-\iiint \mathrm{S}\,\delta\mathbf{H}_{\mathrm{H}}\nabla x\,d\varsigma$, we get, by equating to zero the coefficient of the arbitrary vector $\delta\mathbf{H}$,

$$4\pi_{\mathrm{H}}\nabla x = \mathrm{V}\nabla\mathbf{a} = \mathrm{b} \qquad (26)$$

$$[\mathrm{VU}\nu\mathbf{a}]_{a+b} = 0 \qquad (27),$$

$\mathbf{a}_s$ disappearing, since $[\mathbf{a}_s]_a = [\mathbf{a}_s]_b$.

Again, before considering what is contributed to the left of eq. (24) by $\delta\mathbf{C}$, it must be remembered that $\delta\mathbf{C}$ is not quite arbitrary, by reason of the equations of condition $\mathrm{S}\nabla\mathbf{C} = 0$, $[\mathrm{SU}\nu\mathbf{C}]_{a+b} = 0$. This is taken account of by adding to the left of* eq. (24)

$$\iiint \mathrm{YS}\nabla\,\delta\mathbf{C}\,d\varsigma + \iint \mathrm{Y}_s\mathrm{S}\,d\Sigma\,\delta\mathbf{C} = -\iiint \mathrm{S}\,\delta\mathbf{C}\nabla\mathrm{Y}\,d\varsigma + \iint (\mathrm{Y} + \mathrm{Y}_s)\,\mathrm{S}\,d\Sigma\,\delta\mathbf{C}$$

* It may be objected that these equations of condition have already been taken account of in the treatment accorded to the more general equations of $4\pi\mathbf{C} = \mathrm{V}\nabla\mathbf{H}$, $[\mathrm{VU}\nu\mathbf{H}]_{a+b} = 0$, and, therefore, it is *erroneous to take account of them again.* The answer to this is that it is not *necessary* to do this, but, on the other hand, it is not *erroneous.* We must expect as the result that the Y's will be, in a mathematical sense, redundant. That this actually is the case will appear in § 65 below. The reason for introducing them is to obtain the equations of the field in as familiar a form as possible, and to show the mathematical dependence of the existence of a potential on the equations $\mathrm{S}\nabla\mathbf{C} = 0$, $[\mathrm{SU}\nu\mathbf{C}]_{a+b} = 0$. The process may be paralleled in the subject of the Calculus of Variations. U, V, W being three functions of $x, y, \ldots, \delta x, \delta y, \ldots$, linear in the latter group, let it be required to satisfy the equation $\mathrm{U} = 0$ subject to the equations of condition $\mathrm{V} = 0$, $\mathrm{W} = 0$. The recognised method is to use the single equation $\mathrm{U} + \mathrm{AV} + \mathrm{BW} = 0$ instead of the three, A and B being functions of $x, y, \ldots$ determinable by the problem in hand. It would not be erroneous to add to the left of the last equation CW, where C was a function of the same kind as A and B. One of the two, B or C, would be mathematically redundant, but it might be convenient to introduce both and give arbitrarily, later on, some method of assigning a definite meaning to each.

[eq. (4), § 5], where Y, Y_s are scalars, and, as with $\mathbf{a}_s$, $[Y_s]_a = [Y_s]_b$. Equating now to zero the coefficients of $\delta\mathbf{C}$, $\delta\mathbf{c}$, we get

$$\mathbf{E} = {}_{\mathrm{C}}\nabla x + \mathbf{a} + \nabla \mathrm{Y}, \quad \mathbf{e} = {}_{\mathrm{c}}\nabla x \quad \ldots \ldots \quad (28),$$

$$\mathbf{E}_s = -[\mathrm{YU}\nu]_{a+b}, \quad \mathbf{e}_s = 0 \quad \ldots \ldots \quad (29).$$

It should be noticed that $\mathbf{b}$, defined by eq. (26), satisfies both the conditions of incompressibility

$$\mathrm{S}\nabla\mathbf{b} = 0, \quad [\mathrm{SU}\nu\mathbf{b}]_{a+b} = 0 \quad \ldots \ldots \quad (30).$$

The first condition is obvious from the equation $\mathbf{b} = \mathrm{V}\nabla\mathbf{a}$. The second is easily deduced from the equation $[\mathrm{VU}\nu\mathbf{a}]_{a+b} = 0$. For this last asserts that the component of $\mathbf{a}$ parallel to the surface is the same for both regions bounded. Thus the line integral $\int \mathrm{S}\, d\rho\mathbf{a}$, which, by eq. (3), § 5, $= \iint \mathrm{S}\mathbf{b}\, d\Sigma$, taken over any closed curve on the surface, is the same for both regions. It follows that $[\mathrm{S}\, d\Sigma\mathbf{b}]_{a+b} = 0$. We naturally assume that $\mathbf{a}$ is an intensity and $\mathbf{b}$ a flux. Hence, by § 8,

$$\mathrm{V}\nabla'\mathbf{a}' = \mathbf{b}' \ , \ [\mathrm{VU}\nu'\mathbf{a}']_{a+b} = 0 \quad \ldots \ldots \quad (31)$$

$$\mathrm{S}\nabla'\mathbf{b}' = 0 \ , \ [\mathrm{SU}\nu'\mathbf{b}']_{a+b} = 0 \quad \ldots \ldots \quad (32).$$

39. Next suppose that the only variation implied in equation (24) is in $\dot{\rho}'$, and, therefore, in $\dot{\Psi}$. Thus

$$\iiint \theta\delta(x/\theta)\, d\varsigma = -\iiint \mathrm{S}\delta\dot{\Psi}\zeta_{\dot{\Psi}}\mathrm{C\!\!\!I}x\zeta d\varsigma \text{ [eq. (13) of former paper]}$$

$$= -\tfrac{1}{2}\iiint \mathrm{S}\delta\dot{\Psi}\zeta\chi^{-1}\Phi'\chi'^{-1}\zeta d\varsigma',$$

where Φ' is defined by saying that

$$\Phi = 2_{\dot{\Psi}}\mathrm{C\!\!\!I}x \quad \ldots \ldots \quad (33),$$

and that Φ is a function of Class I. of § 9 above.* Now since [former paper, eq. (39)] $\Psi = \chi'\chi$, we have

$$\delta\dot{\Psi} = \delta\dot{\chi}'.\chi + \chi'\delta\dot{\chi}$$

* What immediately follows is a particular case of a theorem required more than once below. Let Ω, χ and σ be as usual in this paper and let $\mathrm{Q}\omega = -\Sigma\beta\mathrm{S}\omega\sigma$. Then

$$\mathrm{S}\chi'\mathrm{Q}\zeta\Omega\zeta d\varsigma = \Sigma\mathrm{S}\beta\Omega'\sigma' d\varsigma'.$$

More generally, if (ω, ω') be any function of two vectors ω, ω' linear in each

$$(\mathrm{Q}\zeta, \chi\Omega\zeta)\, d\varsigma = \Sigma(\beta, \Omega'\sigma').\, d\varsigma'.$$

and, therefore,

$$S\delta\dot{\Psi}\zeta\chi^{-1}\Phi'\chi'^{-1}\zeta = S\delta\dot{\chi}'\chi\zeta\chi^{-1}\Phi'\chi'^{-1}\zeta + S\chi'\delta\dot{\chi}\zeta\chi^{-1}\Phi'\chi'^{-1}\zeta$$
$$= 2S\delta\dot{\chi}\zeta\Phi'\chi'^{-1}\zeta \text{ [\textit{ibid.}, eq. (6)]}$$

and $\delta\dot{\chi}\omega = - S\omega\nabla . \delta\dot{\rho}'$ [*ibid.*, eq. (25)]. Hence [*ibid.*, eq. (7)]

$$\iiint \theta\delta(x/\theta)d\varsigma = -\iiint S\delta\dot{\rho}'_1\Phi'\chi'^{-1}\nabla_1 d\varsigma'$$
$$= -\iiint S\delta\dot{\rho}_1'\Phi'\nabla_1' d\varsigma' \text{ [\textit{ibid.}, eq. (27)]}$$
$$= -\iint S\delta\dot{\rho}'\Phi' d\Sigma' + \iiint S\delta\dot{\rho}'\Phi'_1\nabla_1' d\varsigma' \text{ [eq. (4), § 5, above].}$$

Hence, from equations (24) (25), above,

$$\mathbf{F} = \Phi'\Delta' \ , \ \mathbf{F}_s = -[\Phi'U\nu']_{a+b} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (34),$$

showing [*ibid.*, p. 107] that the presence in x of $\dot{\Psi}$ leads to a stress Φ.

40. Now, suppose the only variation of eq. (24) is that of temperature. In this case

$$\theta\left(\delta\frac{x}{\theta} + \frac{\partial(\xi/\theta)}{\partial\theta}\delta\theta\right) = \theta\frac{\partial}{\partial\theta}\frac{x+\xi}{\theta}\delta\theta - S\,\delta\Theta_{\circ}\nabla x$$
$$= -\frac{x+\xi}{\theta}\delta\theta - S\nabla\,\delta\theta_{\circ}\nabla x,$$

since [ROUTH's 'El. Rig. Dyn.,' 4th ed., § 410], $\partial(x+\xi)/\partial\theta = 0$. Also [§ 5, eq. (4) above]

$$-\iiint S\nabla\,\delta\theta_{\circ}\nabla x\,d\varsigma = -\iint \delta\theta S\,d\Sigma_{\circ}\nabla x + \iiint \delta\theta S\nabla_{\circ}\nabla x\,d\varsigma$$

Hence, the variation of θ leads to

$$\dot{j} = (x+\xi)/\theta - S\nabla_{\circ}\nabla x \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (35),$$

$$\dot{f}_s = [SU\nu_{\circ}\nabla x]_{a+b} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (36).$$

41. The first of the three statements in § 37 is now obvious, as far as $\mathfrak{c}$ is concerned. With regard to $\mathfrak{C}$ it must be remembered that $\mathfrak{C}$ cannot be made to vary without varying $\mathbf{H}$. Now [RAYLEIGH's 'Sound,' 1st ed., I., § 81] in order that frictional forces may be explained by a dissipation function X, in Lord RAYLEIGH's sense, the frictional force Q corresponding to an independent coordinate q should be $= -\partial X/\partial\dot{q}$. For our purposes this is put more conveniently by saying that $\Sigma Q\,\delta\dot{q} = -\Sigma\,\delta_{\dot{q}}X$,

where the Σ implies that *any assigned group of independent velocities*, and no others, are varied, and where $\delta_{\dot{q}}X$ is the increment in X due to the particular variation $\delta\dot{q}$. Now, on account of the conditions,

$$4\pi\mathbf{C} = \mathrm{V}\nabla\mathbf{H},\ [\mathrm{V}d\Sigma\mathbf{H}]_{a+b} = 0,\ \mathrm{S}\nabla\mathbf{C} = 0,\ [\mathrm{S}d\Sigma\mathbf{C}]_{a+b} = 0,$$

it is necessary that we consider the whole group of velocities, $\delta\mathbf{C}$, throughout space together. It is, then, as far as $\mathbf{C}$ is concerned, only necessary to prove that

$$\iiint \mathrm{S}\mathbf{E}\,\delta\mathbf{C}\,d\varsigma + \iint \mathrm{S}\mathbf{E}_s\,\delta\mathbf{C}\,ds = \iiint (\mathrm{S}\delta\mathbf{C}_{\mathbf{C}}\nabla x + \mathrm{S}\delta\mathbf{H}_{\mathbf{H}}\nabla x)\,d\varsigma,$$

the integrals extending throughout space. (As to the sign of these terms, it must be remembered that the force corresponding to $\mathbf{C}$ is not $\mathbf{E}$, but $-\mathbf{E}$). This is proved quite easily* by means of eq. (4) § 5.

Similarly, with regard to $\dot{\Psi}$, it is only necessary to prove that

$$-\iiint \mathrm{S}\mathbf{F}\,\delta\rho'\,d\varsigma' - \iint \mathrm{S}\mathbf{F}_s\,\delta\dot{\rho}'\,ds' = \iiint \mathrm{S}\,\delta\dot{\Psi}\zeta_{\dot{\Psi}}\mho x\zeta\,d\varsigma,$$

and this is obvious from the mode in which equations (33) (34) were established.

42. To prove the second and third statements, let for any finite region $\iint_b$ denote an integration taken over the *true boundary* of that region, and $\iint_d$ an integral taken over both sides of any surface of discontinuity, as to physical quantities in the region, so that

$$\iint = \iint_b + \iint_d \qquad (37).$$

Then, if we can prove that for *any* finite region,

$$\left.\begin{array}{l}(\text{Rate of increase of heat} + \text{rate of doing work of frictional forces})\\ \quad = -\iint_b \{\mathrm{S}d\Sigma\,(\theta_{\circleddash}\nabla x + \mathrm{V}\mathbf{a}\mathbf{H}/4\pi - \mathrm{Y}\mathbf{C}) + \mathrm{S}\dot{\rho}'\Phi' d\Sigma'\}\end{array}\right\} \qquad (38),$$

it will follow that the energy supply required to account for (1) the increment of heat, (2) the work (negative) done by the frictional forces, consists of three parts, (1)

* It should, perhaps, be noticed that $\delta\mathbf{C}$ and $\delta\mathbf{H}$ are now not perfectly arbitrary. We may assume that

$$\mathrm{S}\nabla\,\delta\mathbf{C} = 0\ ,\ 4\pi\,\delta\mathbf{C} = \mathrm{V}\nabla\,\delta\mathbf{H},$$

and from the equations $[\mathrm{VU}\nu\mathbf{a}]_{a+b} = 0$, $[\mathrm{VU}\nu\mathbf{H}]_{a+b} = 0$

$$[\mathrm{SaU}\nu\,\delta\mathbf{H}]_{a+b} = 0.$$

the work done on the boundary by the viscosity stress, (2) the work done on the boundary by the frictional electric forces, (3) a flux $-\theta_{\circledcirc}\nabla x$ of energy at every point of space. This may be put in, perhaps, the more familiar form :—the increment of heat in the region consists of three parts, (1) the work done *against* the frictional forces throughout the region, (2) the work done *by* the frictional forces (viscosity and electric) on the boundary, and (3) the surface integral taken inwards at the boundary of a flux $-\theta_{\circledcirc}\nabla x$. Stated in this way we see that equation (38) is equivalent to saying that the conduction of heat is due to a flux of heat $-\theta_{\circledcirc}\nabla x$ at every point of space, and that the frictional forces are sources of heat.* These are statements (2) and (3) of § 37 (except that here we have $-\theta_{\circledcirc}\nabla x$, and there we have a more definite form for flux of heat due to conduction.)

To prove eq. (38), note that the expression on the left

$$=\iiint\{(\theta\dot{f}+\mathbf{S}\mathbf{E}\mathbf{C}+\mathrm{S}\mathbf{e}\mathbf{c})\,d\varsigma-\mathrm{S}\mathbf{F}\dot{\rho}'\,d\varsigma'\}+\iint_d\{(\theta\dot{f}_s+\mathrm{S}\mathbf{E}_s\mathbf{C}+\mathrm{S}\mathbf{e}_s\mathbf{c})\,ds-\mathrm{S}\mathbf{F}_s\dot{\rho}'ds'\}$$

$$=\iiint\{[x+\xi-\theta\mathrm{S}\nabla_{\circledcirc}\nabla x+\mathrm{S}\mathbf{C}\,({}_{\mathrm{C}}\nabla x+\mathbf{a}+\nabla\mathrm{Y})+\mathrm{S}\mathbf{c}_{\mathrm{C}}\nabla x]\,d\varsigma-\mathrm{S}\dot{\rho}'\Phi'_1\nabla'_1\,d\varsigma'\}$$

$$+\iint_d\{\mathrm{S}d\Sigma\,(\theta_{\circledcirc}\nabla x+\mathrm{V}\mathbf{a}\mathbf{H}/4\pi-\mathrm{Y}\mathbf{C})+\mathrm{S}\dot{\rho}'\Phi'\,d\Sigma'\}.\dagger$$

Now put $\iint_d=\iint-\iint_b$ [equation (37)], and transform the integral $\iint$ by means of equation (4) § 5 above into a volume integral. In doing this note that by reversing the process of § 39 we get

$$-\iiint\mathrm{S}\dot{\rho}'\Phi'_1\nabla'_1\,d\varsigma'+\iint\mathrm{S}\dot{\rho}'\Phi'\,d\Sigma'=\iiint\mathrm{S}\dot{\Psi}\zeta_{\Psi}\text{Ɑ}x\zeta\,d\varsigma.$$

* It is possible at this stage that two objections may be taken to this reasoning. First it may be said that there ought to be no terms in the surface integral leading to the result that the frictional electric forces do work on the boundary. That this is not a sound objection will come out more clearly below, when the effect of **YC** will be found to in no way alter the ordinary views of the transference of electric energy through the field, and the effect of **VaH** will be only to modify them in a way which would naturally be anticipated from the new hypothesis that **H** has some influence on the frictional forces of the field. Secondly, it may be said that besides the three terms mentioned in the text as contributing to rate of increase of heat, there should be a fourth due to such causes as the THOMSON and PELTIER effects. This statement is, however, undoubtedly wrong, as will appear more clearly when we come to the consideration of these effects. The explanation is that these effects are explained by terms in f. Hence, in equation (38) they are *included* on the left. If this is not considered convincing, let me call attention to equation (25), § 49 below, which asserts that the rate of increase of *intrinsic energy* (including that of the THOMSON effect, &c.), in any space = rate of doing work throughout the region of the external forces which are *not* due to friction + the rate of heat supply from external sources situated in the region + such a surface integral as now is under consideration (*i.e.*, confined to the true boundary).

† We have here for the sake of the next transformation added the term $\iint_d\mathrm{S}d\Sigma\mathbf{a}\mathbf{H}/4\pi$, since from the equations $[\mathrm{VU}\nu\mathbf{H}]_{a+b}=0$, $[\mathrm{VU}\nu\mathbf{a}]_{a+b}=0$, it follows that $[\mathrm{SU}\nu\mathbf{a}\mathbf{H}]_{a+b}=0$.

Thus we get for the expression on the left of eq. (38)

$$\iiint\{x + \xi + S\Theta_{\Theta}\nabla x + S\mathbf{C}_{\mathbf{C}}\nabla x + S\mathbf{c}_{\mathbf{c}}\nabla x + S\mathbf{H}_{\mathbf{H}}\nabla x + S\dot{\Psi}\zeta_{\dot{\Psi}}\mathrm{C}x\zeta\}\, d\varsigma$$
$$- \iint_b\{Sd\Sigma\,(\theta_{\Theta}\nabla x + V\mathbf{a}\mathbf{H}/4\pi - \mathbf{YC}) + S\dot{\rho}'\Phi'\,d\Sigma'\}$$

of which the volume integral is zero [equations (18), (21)], and the surface integral is the expression on the right of equation (38).

To get the ordinary expression for the flux of heat due to conduction we have merely to suppose x to contain the term $- S\Theta\gamma\Theta/2\theta$, where γ is a self-conjugate linear vector function of Class I., of § 9 above. The heat flux referred to the standard position of matter due to this term

$$= \theta_{\Theta}\nabla\,(S\Theta\gamma\Theta/2\theta) = -\,\gamma\Theta,$$

and, therefore, by Prop. VI., § 10, the actual flux of heat is $-\gamma'\Theta'$.

43. It is known (TAIT's 'Heat,' 1st ed., § 412) that if θ_0 be the lowest available temperature, $\theta_0\,\dot{\mathrm{F}}$ is the rate of dissipation or degradation of energy in Sir WILLIAM THOMSON's sense. Now by equations (35), (36),

$$\dot{\mathrm{F}} = \iiint(x + \xi)/\theta\, d\varsigma \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (39),$$

so that $(x + \xi)\,\theta_0/\theta$ may be called the rate of dissipation of energy per unit volume. There seems very good reason then to call X the dissipation function. It only differs from Lord RAYLEIGH's function in the terms that lead to the conduction of heat.

If, as will usually be the case, x is quadratic in Θ and the velocities, $\xi = x$, and the rate of dissipation per unit of volume will be $2x\theta_0/\theta$. For instance, the rate of dissipation per unit volume of the standard position due to conduction $= - S\Theta\gamma\Theta\theta_0/\theta^2$, and, therefore, per unit volume of the present position it is $- S\Theta'\gamma'\Theta'\theta_0/\theta^2$.*

III. ESTABLISHMENT OF GENERAL RESULTS.

A. *Value of* δL *for a Finite portion of Matter.*

44. As already remarked (§ 34) the δ in equation (1) § 13 implies variation in everything but the temperature. This will be assumed for the present. Thus δl depends [§ 27 (25)] on the variations of

$$\rho',\ \dot{\rho}',\ \Psi\ ;\ \mathbf{d},\ \mathbf{D},\ \mathbf{C},\ \mathbf{H}.$$

* I suppose this result has been noticed before, though I do not know by whom.

So far as it depends on ρ', l will be supposed only to contain the term $-\mathrm{D}_m\mathrm{W}$, where D_m is the density of matter in the standard position, and W an ordinary potential (quite independent, however, of electromagnetic phenomena). Of course, so far as it depends on $\dot{\rho}'$, l is supposed only to contain the term $-\mathrm{D}_m\dot{\rho}'^2/2$. Thus δl consists of the following parts :—

$$-\mathrm{S}\,\delta\rho'\nabla' l = \mathrm{D}_m\mathrm{S}\,\delta\rho'\nabla'\mathrm{W} \quad (1),$$

$$-\mathrm{S}\,\delta\dot{\rho}'_{\dot{\rho}'}\nabla l = -\mathrm{D}_m\mathrm{S}\dot{\rho}'\,\delta\dot{\rho}' = -d(\mathrm{D}_m\mathrm{S}\dot{\rho}'\,\delta\rho')/dt + \mathrm{D}_m\mathrm{S}\ddot{\rho}'\,\delta\rho' \quad (2),$$

$$-\mathrm{S}\,\delta\mathbf{D}_{\mathrm{D}}\nabla l \quad (3),$$

$$-\mathrm{S}\,\delta\mathbf{d}_{\mathrm{d}}\nabla l \quad (4),$$

$$-\mathrm{S}\,\delta\mathbf{C}_{\mathrm{C}}\nabla l = -\mathrm{S}\,\delta\dot{\mathbf{D}}_{\mathrm{C}}\nabla l = -d\mathrm{S}_{\mathrm{C}}\nabla l\,\delta\mathbf{D}/dt + \mathrm{S}\delta\,\mathbf{D}d_{\mathrm{C}}\nabla l/dt \quad (5),$$

$$-\mathrm{S}\,\delta\mathbf{H}_{\mathrm{H}}\nabla l = -\mathrm{S}\mathbf{B}\,\delta\mathbf{H}/4\pi \quad (6),$$

$$-\mathrm{S}\,\delta\Psi\zeta\mathrm{C\!\!\!I}\,l\zeta = m\mathrm{S}\,\delta\rho'_1\phi'\nabla'_1 \quad (7),$$

where $\mathrm{C\!\!\!I}$ stands, as throughout the present paper it will stand, for ${}_{\Psi}\mathrm{C\!\!\!I}$, and where ϕ' is defined by saying that

$$\phi = -2\mathrm{C\!\!\!I}\,l \quad (8),$$

and that ϕ is of Class I. in § 9 above. The proof of equation (7) is exactly parallel to the treatment of Φ in § 39 above, and, therefore, need not be given here.

45. The part of $\delta\mathrm{L}$ due to (7) is

$$\iiint \mathrm{S}\,\delta\rho'_1\phi'\nabla'_1\,d\varsigma' = -\iiint \mathrm{S}\,\delta\rho'\phi'_1\nabla'_1\,d\varsigma' + \iint \mathrm{S}\,\delta\rho'\phi'\,\delta\Sigma'$$

by eq. (4), § 5, above. The part* due to (6) is [§ 26, eq. (19)]

$$-(4\pi)^{-1}\iiint \mathrm{S}\nabla\mathbf{A}\,\delta\mathbf{H}\,d\varsigma = -(4\pi)^{-1}\iiint \mathrm{S}\mathbf{A}\nabla\,\delta\mathbf{H}\,d\varsigma - (4\pi)^{-1}\iint \mathrm{S}\mathbf{A}\,\delta\mathbf{H}\,d\Sigma.$$

When considering the whole of space this surface integral can be neglected, since by eq. (15), § 25, $[\mathrm{V}\,d\Sigma\mathbf{H}]_{a+b} = 0$, and by eq. (20), § 26, $[\mathrm{V}\,d\Sigma\mathbf{A}]_{a+b} = 0$. If, as for

* This transformation which assumes a fact still to be proved (viz., that $\mathbf{B} = \mathrm{V}\nabla\mathbf{A}$, $[\mathrm{V}\,d\Sigma\mathbf{A}]_{a+b} = 0$) is given, not with the object of determining the equations of motion, in which process this fact will not be assumed, but to find the rate of change of energy in an assigned space.

the present we assume, we are considering that part of L contributed by a finite portion of matter we must retain the part of the surface integral due to the true boundary of the portion ($\iint_b$ of § 42). Thus the part of δL due to (6) is

$$-\iiint \mathrm{S}\mathbf{A}\,\delta\mathbf{C}\,d\varsigma - (4\pi)^{-1}\iint_b \mathrm{S}\mathbf{A}\,\delta\mathbf{H}\,d\Sigma$$
$$= -\frac{d}{dt}\iiint \mathrm{S}\mathbf{A}\,\delta\mathbf{D}\,d\varsigma + \iiint \mathrm{S}\dot{\mathbf{A}}\,\delta\mathbf{D}\,d\varsigma - (4\pi)^{-1}\iint_b \mathrm{S}\mathbf{A}\,\delta\mathbf{H}\,d\Sigma.$$

Collecting terms we have for any finite portion of matter

$$\delta\mathrm{L} = -\frac{d}{dt}\iiint\{\mathrm{D}_m\mathrm{S}\dot{\rho}'\,\delta\rho' + \mathrm{S}\,\delta\mathbf{D}\,({}_{\mathrm{C}}\nabla l + \mathbf{A})\}\,d\varsigma$$
$$+ \iiint\{\mathrm{S}\,\delta\rho'[\mathrm{D}_m(\ddot{\rho}' + \nabla'\mathrm{W}) - m\phi'_1\nabla'_1] - \mathrm{S}\,\delta\mathrm{d}\,{}_{\mathrm{d}}\nabla l$$
$$+ \mathrm{S}\,\delta\mathbf{D}\,(d_{\mathrm{C}}\nabla l/dt + \dot{\mathbf{A}} - {}_{\mathrm{D}}\nabla l)\}\,d\varsigma$$
$$+ \iint \mathrm{S}\,\delta\rho'\phi'\,d\Sigma' - (4\pi)^{-1}\iint_b \mathrm{S}\mathbf{A}\,\delta\mathbf{H}\,d\Sigma \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (9).$$

B. *The Free Energy and Rate of Increase of Intrinsic Energy for any Finite portion of Matter.*

46. We now see from the principle enunciated in § 33, above, that the modified kinetic energy for all space $\mathfrak{T}_\infty$ is given by

$$2\mathfrak{T}_\infty = -\iiint\{\mathrm{D}_m\dot{\rho}'^2 + \mathrm{S}\mathbf{C}\,({}_{\mathrm{C}}\nabla l + \mathbf{A})\}\,d\varsigma.$$

Now

$$4\pi\iiint \mathrm{S}\mathbf{C}\mathbf{A}d\varsigma = \iiint \mathrm{S}\mathbf{A}\nabla\mathbf{H}d\varsigma \text{ [§ 25, eq. (13)]}$$
$$= \iiint \mathrm{S}\mathbf{H}\nabla\mathbf{A}d\varsigma + \iint \mathrm{S}\mathbf{H}\mathbf{A}d\Sigma \text{ [§ 5, eq. (4)]}$$
$$= \iiint \mathrm{S}\mathbf{B}\mathbf{H}d\varsigma \text{ [§ 26, eq. (19)]},$$

the surface integral vanishing by § 25, eq. (15) and § 26, eq. (20). Thus

$$2\mathfrak{T}_\infty = -\iiint\{\mathrm{D}_m\dot{\rho}'^2 + \mathrm{S}\mathbf{C}_{\mathrm{C}}\nabla l + \mathrm{S}\mathbf{B}\mathbf{H}/4\pi\}\,d\varsigma = \iiint(l + \lambda)\,d\varsigma \quad . \quad . \quad (10),$$

where λ is in value, but not in form (since we suppose it expressed in terms of the same independent variables as l) equal to the function which is reciprocal to l with regard to $\dot{\rho}'$, $\mathbf{C}$ and $\mathbf{H}$. These three vectors may be [§ 35] called velocities, and thus λ is in value the reciprocal of l with regard to all the velocities involved in the latter. Adopting now the notation of § 33 and its *assumption* (end of § 33), we have

$$l + \lambda = 2\mathfrak{t} = -(\mathrm{D}_m\dot{\rho}'^2 + \mathrm{S}\mathbf{C}_\mathrm{C}\nabla l + \mathrm{S}\mathbf{BH}/4\pi) \quad . \quad . \quad . \quad . \quad . \quad (11)$$

$$\lambda_s = \mathfrak{t}_s = 0 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (12).$$

That l, the Lagrangian function (per unit volume), and λ, the free energy, should be reciprocal functions (in value only) with regard to the velocities they contain, is in accord with the fact (but not deducible from it) that a similar statement is true for an ordinary dynamical system [§ 33, eq. (1) above].

Let now Λ stand for the part of the free energy due to a finite portion of matter. Required $\dot{\Lambda}$. To find this, first obtain the rate of increase of Λ that would occur if all the circumstances were such as actually occur, except that the temperature of each element of matter is kept constant, and then add the part due to the rate of variation of temperature. To get the first of these we have at first to find the corresponding part of $\dot{\mathrm{L}}$ by changing all the δ's of eq. (9) into differentiations with regard to the time. Then we have to subtract the result from $\dot{\Lambda} + \dot{\mathrm{L}}$, which is given by eq. (11). Thus we get

$$\left.\begin{aligned} \dot{\Lambda} = & \iiint (\dot{\theta}\,\partial\lambda/\partial\theta - \mathrm{S}\dot{\Theta}_\Theta\nabla\lambda)\,d\varsigma \\ & - \iiint \{\mathrm{S}\dot{\rho}'\,[\mathrm{D}_m(\ddot{\rho}' + \nabla'\mathrm{W}) - m\phi_1'\nabla_1'] - \mathrm{S}\mathbf{c}_d\nabla l + \mathrm{S}\mathbf{C}\,(d_\mathrm{C}\nabla l/dt + \dot{\mathbf{A}} - {}_\mathrm{D}\nabla l)\}\,d\varsigma \\ & - \iint \mathrm{S}\dot{\rho}'\phi'\,d\Sigma' - (4\pi)^{-1}\iint_b \mathrm{S}\dot{\mathbf{A}}\mathbf{H}\,d\Sigma \end{aligned}\right\} (13).$$

It should be shown perhaps how the last integral appears. It comes from the term $-\mathrm{S}\mathbf{BH}/4\pi$ in $l + \lambda$ and from the two terms

$$-\frac{d}{dt}\iiint \mathrm{S}\mathbf{A}\,\delta\mathbf{D}\,d\varsigma - \frac{1}{4\pi}\iint_b \mathrm{S}\mathbf{A}\,\delta\mathbf{H}\,d\Sigma$$

in $\delta\mathrm{L}$. These three terms contribute to $\dot{\Lambda}$

$$\frac{d}{dt}\iiint (\mathrm{S}\mathbf{AC} - \mathrm{S}\mathbf{BH}/4\pi)\,d\varsigma + \frac{1}{4\pi}\iint_b \mathrm{S}\mathbf{A}\dot{\mathbf{H}}\,d\Sigma.$$

But, since, $4\pi\mathbf{C} = \mathrm{V}\nabla\mathbf{H}$, $4\pi\mathrm{S}\mathbf{AC} - \mathrm{S}\mathbf{BH} = \mathrm{S}\mathbf{A}\Delta\mathbf{H}$, so that the volume integral can be transformed into a surface integral. Transforming and noticing that the part of the surface integral $\iint_d$ is zero, the last term in equation (13) is obtained.

We may now obtain $\dot{\mathrm{E}}$. To do this, combine the first integral of equation (13) with $d\,(\iiint \theta f\,d\varsigma + \iint \theta f_s\,ds)/dt$. We thus get by equations (11), (12) of § 34, and equation (4), § 5

$$\iiint \theta\dot{f}\,d\varsigma + \iint \theta\dot{f}_s\,ds - \iint_b \dot{\theta}\mathrm{S}\,d\Sigma_{\circledcirc}\nabla\lambda.$$

Thus, from eq. (13), § 34,

$$\left.\begin{aligned}\dot{\mathrm{E}} = \iiint \theta\dot{f}\,d\varsigma + \iint \theta\dot{f}_s\,ds - \iiint\{&\mathrm{S}\dot{\rho}'\,[\mathrm{D}_m(\ddot{\rho}' + \nabla'\mathrm{W}) - m\phi_1'\nabla_1'] \\ &- \mathrm{S}\mathbf{c}_d\nabla l + \mathrm{S}\mathbf{C}\,(d_{\mathrm{C}}\nabla l/dt + \dot{\mathbf{A}} - {}_{\mathrm{D}}\nabla l)\}\,d\varsigma \\ - \iint \mathrm{S}\dot{\rho}'\phi'\,d\Sigma' - \iint {}_b\mathrm{S}d\Sigma\,(&\mathrm{V}\dot{\mathbf{A}}\mathbf{H}/4\pi + \dot{\theta}_{\circledcirc}\nabla\lambda)\end{aligned}\right\} \quad . \quad (14).$$

C. *The Equations of Motion.*

47. The symbols **E**, **F**, Φ, &c., will now again be supposed to stand for the whole external forces including those due to friction. The parts contributed by all of these except Φ to $\Sigma\mathrm{Q}\,\delta q$ can be written down at once. By the former paper p. 107 the force per unit volume (of present position of matter) due to Φ is $\Phi'\Delta'$. Φ is assumed to be self-conjugate* and of Class I of § 9 above. Thus Φ' is also self-conjugate, and therefore there is, due to it, no couple per unit volume. The force per unit surface at a surface of discontinuity is $-[\Phi'\mathrm{U}\nu']_{a+b}$. Thus the part contributed to $\Sigma\mathrm{Q}\,\delta q$ by Φ is

$$\iint \mathrm{S}\,\delta\rho'\Phi'\,d\Sigma' - \iiint \mathrm{S}\,\delta\rho'\Phi_1'\nabla_1'\,d\varsigma'.$$

Hence collecting all the terms

$$\left.\begin{aligned}\Sigma\mathrm{Q}\,\delta q = -\iiint \mathrm{S}\,\delta\rho'\,(\mathbf{F} + \Phi_1'\nabla_1')\,d\varsigma' - \iint \mathrm{S}\,\delta\rho'\,(\mathbf{F}_s\,ds' - \Phi'\,d\Sigma') \\ + \iiint (\mathrm{S}\mathbf{e}\,\delta\mathbf{d} + \mathrm{S}\mathbf{E}\,\delta\mathbf{D})\,d\varsigma + \iint (\mathrm{S}\mathbf{e}_s\,\delta\mathbf{d} + \mathrm{S}\mathbf{E}_s\,\delta\mathbf{D})\,ds\end{aligned}\right\} \quad . \quad (15).$$

48. To obtain the equations of motion from these results, it must be remembered (§ 38) that while $\delta\rho'$ and $\delta\mathbf{d}$ are quite arbitrary, this is not the case with $\delta\mathbf{D}$ and $\delta\mathbf{H}$. We adopt the same method here as in § 38, *i.e.*, we add to the $\delta\mathrm{L}$ for all space

* It is clear by the work in the former paper (pp. 106 to 108) that there is no *necessity* to make this simplification. On the other hand nothing seems gained by not making it.

$$-\iiint y S\nabla\,\delta\mathbf{D}\,d\varsigma - \iint y_s S\,d\Sigma\,\delta\mathbf{D} - \iiint S\mathbf{A}\,(\delta\mathbf{C} - \nabla\delta\mathbf{H}/4\pi)\,d\varsigma + (4\pi)^{-1}\iint S\mathbf{A}_s\,d\Sigma\,\delta\mathbf{H},$$

where

$$[y_s]_a = [y_s]_b, \qquad [\mathbf{A}_s]_a = [\mathbf{A}_s]_b,$$

and where y, y_s are scalars and $\mathbf{A}$, $\mathbf{A}_s$ vectors ; $\delta\mathbf{D}$ and $\delta\mathbf{H}$ may then both be regarded as arbitrary. The expression to be added to δL may, by equation (4) § 5, be written :—

$$-\frac{d}{dt}\iiint S\mathbf{A}\,\delta\mathbf{D}\,\delta\varsigma + \iiint \{S\,\delta\mathbf{D}\,(\dot{\mathbf{A}} + \nabla y) + S\,\delta\mathbf{H}\nabla\mathbf{A}/4\pi\}\,d\varsigma$$
$$- \iint \{(y + y_s)\,S\,\delta\mathbf{D}\,\delta\Sigma - S\,\delta\mathbf{H}\,(\mathbf{A} + \mathbf{A}_s)\,d\Sigma/4\pi\} \quad . \quad . \quad . \quad (16).$$

Equating now to zero, the coefficient of $\delta\mathbf{H}$ in the extended δL, we get

$$\mathbf{B} = V\nabla\mathbf{A}, \qquad [V\,d\Sigma\mathbf{A}]_{a+b} = 0,$$

the $\mathbf{A}_s$ disappearing on account of the relation $[\mathbf{A}_s]_a = [\mathbf{A}_s]_b$. This is the promised proof of equations (19) (20) of § 26, and, therefore, also of equation (18) of the same article.

49. δL and $\Sigma Q\,\delta q$ are, [(9), (15), (16)], now in such a form that the consequences of equation (1), § 13, are seen by inspection. They give (writing D'_m for D_m/m, so that D'_m is the density of matter in the present position)

$$D'_m\ddot{\rho}' = -D'_m\nabla'W + (\phi' + \Phi')_1\nabla'_1 + \mathbf{F} \quad . \quad . \quad . \quad . \quad . \quad . \quad (17),$$

$$0 = -[(\phi' + \Phi')\,U\nu']_{a+b} + \mathbf{F}_s \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (18),$$

$$\mathbf{e} = {}_{\mathrm{d}}\nabla l \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (19),$$

$$\mathbf{E} = {}_{\mathrm{D}}\nabla l - d_{\mathrm{C}}\nabla l/dt - \dot{\mathbf{A}} - \nabla y \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (20),$$

$$\mathbf{e}_s = 0 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (21),$$

$$\mathbf{E}_s = [yU\nu]_{a+b} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (22).$$

Let, for a *finite* region

$$P = \left.\begin{array}{l}\text{rate of doing work of external forces} \\ \quad + \text{rate of supply of heat from external sources}\end{array}\right\} \quad . \quad . \quad (23),$$

so that P may be called the "power" of the external forces and heat sources. P may be divided into P_f the part due to the frictional forces, *i.e.*, the expression on the left of eq. (38) § 42, and P_e the power of the really external forces and sources, *i.e.*, forces and sources included neither in l nor x. From the equations of motion just obtained, and from eq. (14) § 46 above,

$$\dot{E} = P + \iiint S\mathbf{C}\nabla y d\varsigma - \iint_d yS\mathbf{C}d\Sigma - \iint_b \{ S\dot{\rho}'\phi' d\Sigma' + Sd\Sigma\,(V\dot{\mathbf{A}}\mathbf{H}/4\pi + \dot{\theta}_{\odot}\nabla\lambda)\}.$$

Now, $S\mathbf{C}\nabla y = S\nabla(\mathbf{C}y)$, since $S\nabla\mathbf{C} = 0$. Hence [eq. (4), § 5]

$$\iiint S\mathbf{C}\nabla y dS - \iint_d yS\mathbf{C}d\Sigma = \iint yS\mathbf{C}d\Sigma - \iint_d yS\mathbf{C}d\Sigma = \iint_b yS\mathbf{C}d\Sigma.$$

Hence

$$\dot{E} = P - \iint_b \{S\dot{\rho}'\phi' d\Sigma' + Sd\Sigma\,(- y\mathbf{C} + V\dot{\mathbf{A}}\mathbf{H}/4\pi + \dot{\theta}_{\odot}\nabla\lambda)\} \quad . \quad . \quad (24).$$

Putting now $P = P_e + P_f$ and substituting the expression on the right of eq. (38), § 42 for P_f, we get

$$\dot{E} = P_e - \iint_b \{S\dot{\rho}'(\phi' + \Phi'_f)\, d\Sigma' + Sd\Sigma\,[- (y + Y)\,\mathbf{C} + V\,(\dot{\mathbf{A}} + \mathbf{a})\,\mathbf{H}/4\pi + (\dot{\theta}_{\odot}\nabla\lambda + \theta_{\odot}\nabla x)]\} \quad . \quad . \quad . \quad (25),$$

where now Φ_f' has been put for the Φ' of eq. (38), § 42, to distinguish it from the Φ' of equations (26), (27) below.

50. In §§ 38, 39, it will be remembered that **E**, **F**, &c., stood for those parts only of the external forces which were *due to* X. Let, now, these symbols stand for those parts only of the external forces which are *not involved in* X. Thus in equation (20) we must change **E** into $\mathbf{E} + {}_{\mathbf{C}}\nabla x + \mathbf{a} + \nabla Y$ [eq. (28), § 38], and similarly for the rest of the equations of motion. We thus get

$$D_m'\ddot{\rho}' = - D_m'\nabla' W + (\phi' + \Phi_f' + \Phi')_1\nabla_1' + \mathbf{F} \quad . \quad . \quad . \quad (26).$$

$$0 = - [(\phi' + \Phi_f' + \Phi')\,U\nu']_{a+b} + \mathbf{F}_s \quad . \quad . \quad . \quad (27).$$

$$\mathbf{e} = {}_{\mathrm{d}}\nabla l - {}_{\mathrm{c}}\nabla x \quad . \quad . \quad . \quad (28).$$

$$\mathbf{E} = {}_{\mathrm{D}}\nabla l - (d_{\mathrm{C}}\nabla l/dt + {}_{\mathrm{C}}\nabla x) - (d\mathbf{A}/dt + \mathbf{a}) - \nabla\,(y + Y) \quad . \quad . \quad (29).$$

$$\mathbf{e}_s = 0 \quad . \quad . \quad . \quad (30).$$

$$\mathbf{E}_s = [(y + Y)\,U\nu]_{a+b} \quad . \quad . \quad . \quad (31).$$

We collect here, partly for reference, partly to show more clearly the actual stage we have now reached, the other chief equations of the field.

$$\phi = -2\text{Ꮳ}l \quad \ldots \quad (32).$$

$$\Phi_f = 2_{\dot{\ }}\text{Ꮳ}x \quad \ldots \quad (33).$$

$$\left.\begin{aligned} 4\pi_{\text{H}}\nabla l = \mathbf{B} = \text{V}\nabla\mathbf{A}, \quad & [\text{VU}\nu\mathbf{A}]_{a+b} = 0 \\ \text{S}\nabla\mathbf{B} = 0, \quad & [\text{SU}\nu\mathbf{B}]_{a+b} = 0 \end{aligned}\right\} \quad \ldots \quad (34).$$

$$\left.\begin{aligned} 4\pi_{\text{H}}\nabla x = \mathfrak{b} = \text{V}\nabla\mathfrak{a}, \quad & [\text{VU}\nu\mathfrak{a}]_{a+b} = 0 \\ \text{S}\nabla\mathfrak{b} = 0, \quad & [\text{SU}\nu\mathfrak{b}]_{a+b} = 0 \end{aligned}\right\} \quad \ldots \quad (35).$$

$$\left.\begin{aligned} 4\pi\mathbf{C} = \text{V}\nabla\mathbf{H}, \quad & [\text{VU}\nu\mathbf{H}]_{a+b} = 0 \\ \text{S}\nabla\mathbf{C} = 0, \quad & [\text{SU}\nu\mathbf{C}]_{a+b} = 0 \\ \text{S}\nabla\mathbf{D} = 0, \quad & [\text{SU}\nu\mathbf{D}]_{a+b} = 0 \end{aligned}\right\} \quad \ldots \quad (36).$$

$$\left.\begin{aligned} \mathbf{D} = \mathfrak{d} + \mathfrak{k}, \quad & \mathbf{C} = \mathfrak{c} + \mathbf{K} \\ \mathfrak{c} = \dot{\mathfrak{d}}, \quad \mathbf{K} = \dot{\mathfrak{k}}, \quad & \mathbf{C} = \dot{\mathbf{D}} \end{aligned}\right\} \quad \ldots \quad (37).$$

$$\left.\begin{aligned} \mathfrak{c}' = \dot{\mathfrak{d}}' + \text{V}\nabla'_1\text{V}\mathfrak{d}'\dot{\rho}'_1 = \partial\mathfrak{d}'/\partial t + \text{V}\nabla'\text{V}\mathfrak{d}'\dot{\rho}' - \dot{\rho}'\text{S}\nabla'\mathfrak{d}' \\ \mathbf{C}' = \dot{\mathbf{D}}' + \text{V}\nabla'_1\text{V}\mathbf{D}'\dot{\rho}'_1 = \partial\mathbf{D}'/\partial t + \text{V}\nabla'\text{V}\mathbf{D}'\dot{\rho}' \end{aligned}\right\} \quad \ldots \quad (38).$$

[The last set has not yet been proved, as it is more convenient to discuss it along with the detailed results, though clearly itself a general result.] Roughly speaking, of these equations [(25) to (38)], it may be said that (36) and (37) contain the assumptions of the present theory, and the rest the consequences of those assumptions.

Two remarks may be made here. It is clear that, since in the equations (25) to (38), y and Y occur only under the form $y + \text{Y}$, there is nothing by means of which we could experimentally distinguish them. Putting, then,

$$y + \text{Y} = v \quad \ldots \quad (39),$$

we shall generally in the future speak only of v. It may be conveniently called the potential, though, as we shall see later, this is not in accordance with MAXWELL's usage of the term; and, what is perhaps of more importance, there is something arbitrary about it apart from the arbitrary additive constant which every potential involves.

The second thing to notice is that the **E** of equation (29) is not what is usually known as the electromotive force. The physical fact that is usually stated by saying that $\mathbf{E} = \mathbf{RK}$, must with the present notation be stated by saying that $\mathbf{E} = 0$, since $-\mathbf{RK}$ appears on the right of equation (29) as a part of the term $-{}_{\mathrm{C}}\nabla x$. This, of course, is due to the fact that **E** of equation (29) is physically defined as the part of the electromotive force not depending on friction.

D. *Change of Variables in l, λ, and x.*

51. In what follows with reference to change of variables, we shall always speak as if the change had reference only to l. Exactly similar reasoning applies to similar changes of variables in any other function such as λ, x, or a part only of any one of these. There is, indeed, no reason why the function should be a scalar.

So far, l has been assumed an explicit function of the list of variables (25), § 27. These are by far the most convenient variables for most mathematical operations, and we shall continue as often as otherwise so to regard l. For many physical interpretations, however, it is necessary to regard l, or a part of it, expressed in terms of other variables. Consider, for instance, air as a dielectric. This will be taken account of by supposing l to contain a term quadratic in **d**. Suppose, now, we compress the air till its density is (say) doubled. We know as a matter of experimental fact, that the specific inductive capacity will not thereby be largely altered. This will mean, *not* that the quadratic expression in **d** is but slightly altered in form, *but* that the equal expression in **d**′ is thus slightly altered. Moreover, to express simply the fact of electric and magnetic isotropy of fluids requires that the independent variables should be the dashed letters. Let then

$$l d\varsigma = l' d\varsigma' = l'' d\varsigma' \text{ or } l = ml' = ml'' \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (1),$$

where

$$\left.\begin{array}{llll} l & \text{is an explicit function of} & \theta,\ \Theta\ ;\ \rho',\ \dot{\rho}',\ \Psi\ ;\ \mathbf{d},\ \mathbf{D},\ \mathbf{C},\ \mathbf{H} \\ l' & \quad ,, \qquad\qquad ,, & \theta,\ \Theta'\ ;\ \rho',\ \dot{\rho}',\ \Psi\ ;\ \mathbf{d}',\ \mathbf{D}',\ \mathbf{C}',\ \mathbf{H}'\ ;\ q \\ l'' & \quad ,, \qquad\qquad ,, & \theta,\ \Theta''\ ;\ \rho',\ \dot{\rho}',\ \Psi\ ;\ \mathbf{d}'',\ \mathbf{D}'',\ \mathbf{C}'',\ \mathbf{H}'' \end{array}\right\} \quad . \quad (2).$$

Defining λ', λ'' similarly, it may be said here what will appear incidentally later, that λ' and l', and again, λ'' and l'', are related to one another exactly as are λ and l; *i.e.* [§ 46, eq. (11)]

$$\left.\begin{array}{l} l' + \lambda' = -\,\mathrm{D}'_m\dot{\rho}'^2 - \mathrm{S}\mathbf{C}'\,{}_{\mathrm{C}}\nabla' l' - \mathrm{S}\mathbf{H}'\,{}_{\mathrm{H}}\nabla' l' \\ l'' + \lambda'' = -\,\mathrm{D}'_m\dot{\rho}'^2 - \mathrm{S}\mathbf{C}''\,{}_{\mathrm{C}}\nabla'' l'' - \mathrm{S}\mathbf{H}''\,{}_{\mathrm{H}}\nabla'' l'' \end{array}\right\} \quad . \quad . \quad . \quad . \quad (3),$$

where ${}_{\mathrm{C}}\nabla'$ is put for ${}_{\mathrm{C}'}\nabla$, &c.

52. In (2) it is to be noticed that one more variable, viz., q, occurs in l' than in l or l''. The reason is obvious, but on account of the fact, it is easiest to arrive at formulæ transforming differentiations of l into the corresponding ones of l' by first considering the similar relations between l and l''.

Let σ and τ be taken as a typical independent variable intensity and flux respectively. l' is obtained from l'' merely by changing every σ'' and τ'' into $q^{-1}\sigma' q$ and $q^{-1}\tau' q$ respectively. (§ 7.)

By considering the increment in l' and l'' due to an increment in a σ'' or τ'', we at once obtain

$$ {}_\sigma\nabla' l' = q {}_\sigma\nabla'' l'' q^{-1}, \quad {}_\tau\nabla' l' = q {}_\tau\nabla'' l'' q^{-1} \quad \ldots \ldots \quad (4). $$

By a similar process it is easy to see that

$$ \left.\begin{aligned} \frac{\partial l}{\partial \theta} d\varsigma &= \frac{\partial l'}{\partial \theta} d\varsigma' = \frac{\partial l''}{\partial \theta} d\varsigma' \\ {}_{\rho'}\nabla l d\varsigma &= {}_{\rho'}\nabla l' d\varsigma' = {}_{\rho'}\nabla l'' d\varsigma' \\ \nabla' l &= \nabla'(ml') = \nabla'(ml'') \\ \mathrm{C\!l} l' &= \mathrm{C\!l} l'' \end{aligned}\right\} \quad \ldots \ldots \quad (5). $$

53. We proceed to find the corresponding relations for the other variables. Let us in l and l'' vary Ψ and every σ and τ, and σ'' and τ''. Thus,

$$ \begin{aligned} -\Sigma S\delta\sigma {}_\sigma\nabla l - \Sigma S\delta\tau {}_\tau\nabla l - S\delta\Psi\zeta \mathrm{C\!l} l\zeta &= \delta l = l''\delta m + m\delta l'' \\ &= l''\delta m + m\{-\Sigma S\delta\sigma'' {}_\sigma\nabla'' l'' - \Sigma S\delta\tau'' {}_\tau\nabla'' l'' - S\delta\Psi\zeta \mathrm{C\!l} l''\zeta\}. \end{aligned} $$

Now (§ 7)

$$ \tau'' = m^{-1}\psi\tau, \quad \sigma'' = \psi^{-1}\sigma. $$

Hence

$$ \begin{aligned} \delta\tau'' &= m^{-1}(\delta\psi - m^{-1}\delta m . \psi)\tau + m^{-1}\psi\delta\tau. \\ \delta\sigma'' &= -\psi^{-1}\delta\psi\psi^{-1}\sigma + \psi^{-1}\delta\sigma. \end{aligned} $$

Substituting these values and equating the vector coefficients of the arbitrary vectors $\delta\sigma$ and $\delta\tau$, we obtain

$$ {}_\sigma\nabla l = m\psi^{-1} {}_\sigma\nabla'' l'' = m\chi^{-1} {}_\sigma\nabla' l' \quad \ldots \ldots \quad (6), $$

$$ {}_\tau\nabla l = \psi {}_\tau\nabla'' l'' = \chi' {}_\tau\nabla' l' \quad \ldots \ldots \quad (7), $$

the last result in each of these being given by equation (4). These equations show that ${}_\sigma\nabla l$, ${}_\tau\nabla' l'$, ${}_\sigma\nabla'' l''$ bear to one another exactly the same relations as τ, τ', τ'', which may be expressed by saying that they are fluxes.* Similarly, ${}_{\tau}\nabla l$ is an intensity. This particular result can, of course, be proved by a simpler process than the above. We now see that the meaning of $\mathbf{B}'$, obtained by defining $\mathbf{B}$ as a flux, $= 4\pi_{\mathrm{H}}\nabla l$, and likewise the meaning of $\mathfrak{b}'$ is independent of the particular position of matter we take as the standard. We also see similarly that the various terms in $\mathbf{E}'$, $\mathfrak{e}'$, resulting from regarding these vectors as intensities, and utilising equations (28), (29), § 50, will be independent of the particular standard position chosen. And again, by Prop. II., § 8, we now see that equations (3) of last section must be true.

54. Putting now $\delta\sigma = 0$, $\delta\tau = 0$, the equation $\delta l = l''\,\delta m + m\,\delta l''$ gives

$$- \mathrm{S}\,\delta\Psi\zeta\mathrm{Cl}l\zeta = - m\mathrm{S}\,\delta\Psi\zeta\mathrm{Cl}l''\zeta + (l'' + m^{-1}\Sigma\mathrm{S}\psi\tau_{\tau}\nabla''l'')\,\delta m$$
$$+ (m\Sigma\mathrm{S}\psi^{-1}\,\delta\psi\psi^{-1}\sigma_{\sigma}\nabla''l'' - \Sigma\mathrm{S}\,\delta\psi\tau_{\tau}\nabla''l'').$$

Now, by former paper, eq. (18),

$$6m = \mathrm{S}\zeta_1\zeta_2\zeta_3\mathrm{S}\psi\zeta_1\psi\zeta_2\psi\zeta_3.$$

Hence

$$2\,\delta m = \mathrm{S}\,\delta\psi\zeta\mathrm{V}\psi\zeta_1\psi\zeta_2\mathrm{S}\zeta\zeta_1\zeta_2,$$

or, by eq. (10) of former paper,

$$\delta m = - m\mathrm{S}\,\delta\psi\zeta\psi^{-1}\zeta \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (8).$$

Similarly, since m^2 is the same function of Ψ as m is of ψ,

$$\delta m = - \frac{m}{2}\,\mathrm{S}\,\delta\Psi\zeta\Psi^{-1}\zeta \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (9).$$

Also, for future use, note that since $\delta m = - \mathrm{S}\,\delta\psi\zeta_{\psi}\mathrm{Cl}m\zeta = - \mathrm{S}\,\delta\Psi\zeta\mathrm{Cl}m\zeta$ these equations give (former paper, p. 105),

$${}_{\psi}\mathrm{Cl}m = m\psi^{-1}, \quad \mathrm{Cl}m = \tfrac{1}{2}m\Psi^{-1} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (10).$$

* It is interesting to notice a particular result of this. Since Θ is an intensity, ${}_{\Theta}\nabla l$ is a flux. Hence [Prop. IV., § 8] $\mathrm{S}\nabla_{\Theta}\nabla l = m\mathrm{S}\nabla'_{\Theta}\nabla' l'$. Dismissing the particular notation of this paper for the moment, and putting x, y, z for the coordinates of ρ and λ, μ, ν for those of ρ', this may be written

$$m^{-1}\left\{\frac{\partial}{\partial x}\left(\frac{\partial l}{\partial\frac{\partial\theta}{\partial x}}\right) + \frac{\partial}{\partial y}\left(\frac{\partial l}{\partial\frac{\partial\theta}{\partial y}}\right) + \frac{\partial}{\partial z}\left(\frac{\partial l}{\partial\frac{\partial\theta}{\partial z}}\right)\right\} = \frac{\partial}{\partial\lambda}\left(\frac{\partial l'}{\partial\frac{\partial\theta}{\partial\lambda}}\right) + \frac{\partial}{\partial\mu}\left(\frac{\partial l'}{\partial\frac{\partial\theta}{\partial\mu}}\right) + \frac{\partial}{\partial\nu}\left(\frac{\partial l'}{\partial\frac{\partial\theta}{\partial\nu}}\right).$$

If we add [equation (5), § 52, above] $-m^{-1}\partial l/\partial\theta$ to the left of this equation and $-\partial l'/\partial\theta$ to the right, we get a well-known theorem of JACOBI's. Comparing with the form of this theorem given in TODHUNTER's 'History of the Calculus of Variations,' § 323, equation (2), his G, Γ, v, ϕ, Π are our l, ml', θ (regarded as a function of ρ), θ (regarded as a function of ρ'), and m^{-1} respectively. See also TODHUNTER's 'Functions of LAPLACE, LAMÉ, and BESSEL,' § 298, equation (17), and the supplementary volume of BOOLE's 'Differential Equations,' p. 216.

In the equation for δl put $\delta\psi\psi^{-1}\sigma = -\delta\psi\zeta S\zeta\psi^{-1}\sigma$ and $\delta\psi_\tau\nabla''l'' = -\delta\psi\zeta S\zeta_\tau\nabla''l''$; for δm substitute from eq. (9); and for l'', ${}_\tau\nabla''l''$ and ${}_\sigma\nabla''l''$, substitute in terms of l, ${}_\tau\nabla l$ and ${}_\sigma\nabla l$. Thus

$$-S\,\delta\Psi\zeta\mathrm{C}l\zeta = -S\,\delta\Psi\zeta\{m\mathrm{C}l''\zeta + \tfrac{1}{2}(l + \Sigma S\tau_\tau\nabla l)\Psi^{-1}\zeta\},$$
$$-S\,\delta\psi\zeta\{\Sigma_\sigma\nabla l S\zeta\psi^{-1}\sigma - \Sigma\tau S\zeta\psi^{-1}{}_\tau\nabla l\}.$$

Now, let Ω, v be two functions of Class I. of § 9, the first given by

$$\Omega = -2\mathrm{C}l + 2m\mathrm{C}l'' + (l + \Sigma S\tau_\tau\nabla l)\Psi^{-1} \quad . \quad . \quad . \quad . \quad . \quad . \quad (11),$$

from which [eq. (9) § 9, eq. (32) § 50, and Prop. II. § 8]

$$\Omega' = \phi' + 2\chi\mathrm{C}l''\chi' + l' + \Sigma S\tau'_\tau\nabla'l' \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (12).$$

Let v be given by

$$v\omega = \Sigma(\tau S_\tau\nabla l\psi^{-2}\omega - {}_\sigma\nabla l S\sigma\psi^{-2}\omega) \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (13),$$

from which are easily deduced

$$v'\omega = \Sigma(\tau' S_\tau\nabla'l'\omega - {}_\sigma\nabla'l' S\sigma'\omega) \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (14),$$

$$v''\omega = q^{-1}v'(q\omega q^{-1})q = \Sigma(\tau'' S_\tau\nabla''l''\omega - {}_\sigma\nabla''l'' S\sigma''\omega) \quad . \quad . \quad . \quad . \quad (15).$$

From the last value for δl we now have

$$S\,\delta\Psi\zeta\Omega\zeta = 2S\,\delta\psi\zeta v\psi\zeta,$$

or

$$S\,\delta\psi\zeta\Omega\psi\zeta = S\,\delta\psi\zeta v\psi\zeta.$$

Hence (former paper, p. 105) the pure part of $\Omega\psi =$ ditto $v\psi$, *i.e.*,

$$\Omega\psi = v\psi + mV\eta(\ ),$$

where η is a vector to be determined. Hence

$$\Omega\omega = v\omega + mV\eta\psi^{-1}\omega \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (16).$$

Therefore

$$\Omega'\omega = v'\omega + \chi V\eta q^{-1}\omega q \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (17),$$

and

$$\Omega''\omega = v''\omega + \psi V\eta\omega \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (18).$$

From the last equation and the fact that $V\zeta\Omega''\zeta = 0$, it is easy to deduce that

$$\eta = (\psi + S\zeta\psi\zeta)^{-1}\Sigma V(\sigma''{}_\sigma\nabla''l'' + \tau''{}_\tau\nabla''l'') \quad . \quad . \quad . \quad . \quad . \quad . \quad (19),$$

and by taking the pure part of both sides of eq. (17) we get

$$2\Omega'\omega = \Sigma\mathrm{V}(\tau'\omega_\tau\nabla' l' - \sigma'\omega_\sigma\nabla' l') + \omega\Sigma\mathrm{S}(\tau'_\tau\nabla' l' - \sigma'_\sigma\nabla' l') + \chi\mathrm{V}\eta q^{-1}\omega q + q\mathrm{V}\chi'\omega\eta \,.\, q^{-1},$$

whence, putting

$$[\phi'] = -2\chi\mathfrak{C}l''\chi' = -2\chi\mathfrak{C}l'\chi' \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (20),$$

we have from eq. (12)

$$\left.\begin{aligned}2\phi' = 2[\phi'] - \{2l' + \Sigma\mathrm{S}(\tau'_\tau\nabla' l' + \sigma'_\sigma\nabla' l')\} + \Sigma\mathrm{V}\{\tau'(\ \ )_\tau\nabla' l' - \sigma'(\ \ )_\sigma\nabla' l'\} \\ + \{\chi\mathrm{V}\eta q^{-1}(\ \ )q + q\mathrm{V}\chi'(\ \ )\eta \,.\, q^{-1}\}\end{aligned}\right\} \quad (21).$$

Note that the terms here depending on η may be put in the form

$$\chi\mathrm{V}\eta q^{-1}(\ \ )q + q\mathrm{V}\chi'(\ \ )\eta \,.\, q^{-1} = q\varpi q^{-1}(\ \ )q \,.\, q^{-1} \quad . \quad . \quad . \quad (22),$$

where ϖ is the self-conjugate linear vector function given by

$$\varpi = \psi\mathrm{V}\eta(\ \ ) + \mathrm{V}\psi(\ \ )\eta \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (23).$$

For purposes of physical interpretation it is often legitimate to assume the present and standard positions to coincide. In this case $q = 1$, $\chi = \chi' = 1$, so that

$$2\phi' = 2[\phi'] - \{2l + \Sigma\mathrm{S}(\tau_\tau\nabla l + \sigma_\sigma\nabla l)\} + \Sigma\mathrm{V}\{\tau(\ \ )_\tau\nabla l - \sigma(\ \ )_\sigma\nabla l\} \quad (24),$$

and if, further, l is a homogeneous quadratic function of the σ's and τ's,

$$2\phi' = 2[\phi'] + \Sigma\mathrm{V}\{\tau(\ \ )_\tau\nabla l - \sigma(\ \ )_\sigma\nabla l\} \quad . \quad . \quad . \quad . \quad . \quad (25).$$

55. For future use we will make two deductions from these results. First suppose that

$$l = l_0 + m(2\pi\mathrm{K}_0^{-1}\,\mathrm{d}'^2 - \mu_0\mathbf{H}'^2/8\pi) \quad . \quad . \quad . \quad . \quad . \quad . \quad (26),$$

where K_0, μ_0 are absolute constant scalars—the specific inductive capacity and magnetic permeability of a vacuum, and where l_0 is expressed in terms of the undashed letters. Thus it is only in the part of l independent of l_0 that the change of variables is made. In this part there is one τ', viz., d', and one σ', viz., $\mathbf{H}'$; and $\eta = 0$. Hence

$$\phi'\omega = -2m^{-1}\chi\mathfrak{C}l_0\chi'\omega - 2\pi\mathrm{K}_0^{-1}\,\mathrm{d}'\omega\,\mathrm{d}' - \mu_0\mathbf{H}'\omega\mathbf{H}'/8\pi \quad . \quad . \quad . \quad (27).$$

Next let l_0' be what l_0 becomes when expressed in terms of the dashed letters.

Note that l_0' does not stand towards l_0 in the same way as l' towards l, as appears by the equations

$$ml' = l, \qquad l_0' = l_0 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (28).$$

In utilising equation (21), then, the analogue of l' will be $m^{-1}l'_0$. Thus the part contributed by l'_0 to the terms $[\phi'] - l'$ on the right of equation (21) will be

$$-2\chi\text{Ɑ}(m^{-1}l_0')\chi' - m^{-1}l_0' = -2m^{-1}\chi\text{Ɑ}l_0'\chi' + m^{-1}l_0'\chi\Psi^{-1}\chi' - m^{-1}l_0', \text{ [equation (10)]},$$
$$= -2m^{-1}\chi\text{Ɑ}l_0'\chi',$$

since [former paper, equation (39)] $\Psi^{-1} = \chi^{-1}\chi'^{-1}$. Assuming, then, ϕ_0 to be of the first class of § 9, and defined by

$$\phi_0 = -2\text{Ɑ}l_0' \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (29),$$

equation (21) gives

$$\begin{aligned}2\phi' = 2\phi'_0 &+ (\mu_0\mathbf{H}'^2/4\pi - 4\pi\mathrm{K}_0^{-1}\,\mathrm{d}'^2) \\ &- \Sigma\mathrm{S}\,(\tau'_\tau\nabla'l' + \sigma'_\sigma\nabla'l') + \Sigma\mathrm{V}\,\{\tau'(\quad)_\tau\nabla'l' - \sigma'(\quad)_\sigma\nabla'l'\} \\ &+ \{\chi\mathrm{V}\eta q^{-1}(\quad)q + q\mathrm{V}\chi'(\quad)\eta\,.\,q^{-1}\} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (30),\end{aligned}$$

where l', σ', τ', and η have exactly the same meanings as before, so that, indeed,

$$l' \equiv l'_0/m + 2\pi\mathrm{K}_0^{-1}\,\mathrm{d}'^2 - \mu_0\mathbf{H}'^2/8\pi \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (31).$$

E. *Connection between* **E** *and* **e**.

56. So far it has been assumed that there are two independent kinds of external force denoted by **E** and **e**, and by $\mathbf{E}_s$ and $\mathbf{e}_s$. This is contrary to the usual custom, but seems to me to be a necessary consequence of assumptions always made as to the difference in nature between what is ordinarily called the displacement current and the conduction current.

The independent variables required to fix the electric state at a point have for mathematical convenience been taken as **D** and **d**. These are, perhaps, not the most natural. It would seem from the ordinary views as to the two kinds of current as if the dielectric displacement **d**, and the conduction displacement **k** are the most natural. Moreover, I believe it is generally held that **d** has exclusively to do with the potential energy of electrification. It seems, then, likely to lead to correct results to assume that if **d** and **k** were taken as the independent coordinates, there would never be any external force of type **d**.

As this conclusion may seem open to question let us put the matter in a different way. If (regarding **d** and **k** as the independent electric coordinates) we could be certain that we had found the full expressions for l, l_s, x, x_s, *both* types of external electromotive force would be zero. But we can with considerable certainty say that we have not found these completely, so far as they depend upon **k** and **K** (electrolysis, &c.). On the other hand, it is by no means so obvious that we have not found them completely so far as they depend on **d** and **c**. Let us then assume that the external force (exclusive of frictional forces, of course) of the latter type is zero. If we can point to no experimental facts contradicted by this assumption, we may consider that the simplification is warranted.

57. Now (§ 28) the work done per unit volume by the external forces **E**, **e** of equations (28), (29), § 50 above, while **D** and **d** suffer the increments $d\mathbf{D}$ and $d\mathbf{d}$ respectively, is

$$\begin{aligned} \mathrm{S}\mathbf{E}\,d\mathbf{D} + \mathrm{S}\mathbf{e}\,d\mathbf{d} &= \mathrm{S}\mathbf{E}\,(d\mathbf{d} + d\mathbf{k}) + \mathrm{S}\mathbf{e}\,d\mathbf{d} \\ &= \mathrm{S}\,(\mathbf{E} + \mathbf{e})\,d\mathbf{d} + \mathrm{S}\mathbf{E}d\mathbf{k}. \end{aligned}$$

Hence, if **d** and **k** be taken as the coordinates, the forces of those types would be $\mathbf{E} + \mathbf{e}$ and **E** respectively. The assumption just made then leads to

$$\mathbf{E} + \mathbf{e} = 0, \quad \mathbf{E}_s + \mathbf{e}_s = 0 \quad \ldots \ldots \ldots \ldots \quad (1),$$

where it must be remembered that the exact meaning of these four symbols is that given to them in § 50 above, not the meaning they had previous to that section. If we assumed that x was independent of **H**, equation (1) would be equally true of the previous meanings of the symbols.

We shall now always suppose **e** to be replaced by $-\mathbf{E}$. With regard to $\mathbf{e}_s$ and $\mathbf{E}_s$, note that by means of equation (1) and equations (30), (31), and (39) of § 50

$$[v\mathrm{U}\nu]_{a+b} = 0 \quad \ldots \ldots \ldots \ldots \quad (2),$$

which shows that what we have called the potential is continuous throughout space. This will be found to lead to the result that contact-force cannot be explained without a slight extension of the independent variables of l, or the assumption that l_s is not zero. It does not, however, prevent on present assumptions an explanation of the Peltier effect.

IV. Detailed Examination of these Results.

A. *Maxwell's Results.*

58. The justification of the present theory, where it differs from accepted theory, must be based on an examination of its results in detail. First, then, let us compare

with MAXWELL's results. With the exception of (1) the expression for current in terms of displacement for a moving body, and (2) certain of his mechanical results which I hold to be inconsistent with certain others of his own, it will be found that his results flow from the equations now established.

We put down, then, simple forms of l and x, the first involving as independent variables Ψ, $\mathbf{d}$ and $\mathbf{H}$, and the second Ψ and $\mathbf{K}$ only, and compare the results with MAXWELL's. Besides MAXWELL's results we shall find that this form of l is sufficient to take account of the interdependence of magnetisation and strain, and of specific inductive capacity and strain. After that we add certain terms to, and otherwise generalise l and x, still, however, regarding them as involving no independent variables except such as occur in the lists (25), (26), of § 27. Thermoelectric, thermomagnetic, and the HALL phenomena are thereby accounted for and discussed in detail. Finally, to account for electrostatic contact-force (and incidentally capillary phenomena), we shall assume l to contain certain independent variables not in the list (25) of § 27, and shall adopt a certain form for l_s.

59. For MAXWELL's results it is only necessary to assume

$$l = 2\pi \mathrm{S}\,\mathbf{d}\mathrm{K}^{-1}\mathbf{d} - \mathrm{S}\mathbf{H}\mu\mathbf{H}/8\pi - \mathrm{S}\mathbf{I}_0\mathbf{H} \quad \text{(1)},$$

$$x = -\mathrm{S}\mathbf{K}\mathrm{R}\mathbf{K}/2 \quad \text{(2)},$$

where $\mathbf{I}_0$ is a flux, μ and K* are self-conjugate functions of Class I. of § 9, and R is one of Class II, all four being functions of strain and temperature. From these statements, and § 9, it follows that

* I did not notice when first [former paper, p. 119] using K in this signification that it already had a special quaternion meaning (conjugate of a quaternion). As this meaning is never required in the present paper, and very rarely in physical applications, I have nevertheless retained the present meaning for K.

I take this opportunity of apologising for the apparent want of system in my notation. It has been brought about by an attempt to compromise between accepted notation and a system of notation more suitable for quaternion methods. May I suggest the following system? First, let the Greek alphabet be left as a happy hunting ground for symbols of every denomination (vectors, scalars, linear vector functions, &c.); secondly, let the ordinary alphabets, A, B . . ., a, b . . ., be used for scalars and linear vector functions of a vector (which so often in important cases reduce to scalars) only; thirdly, let bold type be used for vectors only and write **i**, **j**, **k** instead of i, j, k; fourthly, let HAMILTON's K, S, T, U, V be transferred to the German alphabet; fifthly, let the rest of the two German alphabets be retained for mathematicians who are hard pressed for suitable symbols; sixthly, let the symbols of differentiation be quite independent of the above restrictions. The following somewhat chaotic but classified list of some of the chief symbols used in the present paper may serve to convince the sceptic that *some such* system is necessary. (1.) *Linear vector functions of a vector* (20), A B, C K, R, a, b, c, r, Υ, Φ, Ψ, Ω, γ, μ, ϖ, v, ϕ, χ, ψ. (2.) *Vectors* (31), i, j, k, **A**, **B**, **C**, **D**, **E**, **F**, **H**, **I**, **K**, **L**, **N**, **P**, **a**, **b**, **c**, **d**, **e**, **h**, Θ, $d\Sigma$, α, ϵ, η, $\mathrm{U}\nu$, ρ, σ, τ, ω. (3.) *Scalars* (32), D, E, F, H, P, Q, W, X, Y, Z f, g, h, l, m, n, q, s, t, u, v, x, y, z, θ, λ, ξ, $\mathfrak{D}$, $\mathfrak{T}$, $\mathfrak{B}$, t. (4.) *Symbols of differentiation and variation*, $\mathfrak{A}$, d, ∂, Δ, ∇, δ, ζ. (5.) *Symbols of peculiar quaternion meaning*, S, T, U, V.

$$\left.\begin{aligned} l' &= 2\pi \mathrm{Sd'K'^{-1}d'} - \mathrm{S}\mathbf{H}'\mu'\mathbf{H}'/8\pi - \mathrm{S}\mathbf{I}'_0\mathbf{H}' \\ l'' &= 2\pi \mathrm{Sd''K''^{-1}d''} - \mathrm{S}\mathbf{H}''\mu''\mathbf{H}''/8\pi - \mathrm{S}\mathbf{I}''_0\mathbf{H}'' \end{aligned}\right\} \quad . \quad . \quad . \quad . \quad . \quad . \quad (3),$$

$$x' = -\mathrm{S}\mathbf{K}'\mathrm{R}'\mathbf{K}'/2, \quad x'' = -\mathrm{S}\mathbf{K}''\mathrm{R}''\mathbf{K}''/2 \quad . \quad . \quad . \quad . \quad . \quad . \quad (4).$$

60. Most of MAXWELL's results are collected together in § 619 of his 'Electricity and Magnetism,' 2nd edition. In our notation they are

$$\mathbf{B}' = \mathrm{V}\nabla'\mathbf{A}' \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (\mathrm{A})\ (5),$$

$$\mathbf{E}'_0 = \mathrm{V}\dot{\rho}'\mathbf{B}' - \partial\mathbf{A}'/\partial t - \nabla' z \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (\mathrm{B})\ (6),$$

where $\partial/\partial t$ denotes differentiation with regard to time at a fixed point of space, and where z is some scalar put for MAXWELL's Ψ. Equation (C) we omit for the present, as it requires more detailed discussion than the others.

$$\mathbf{B}' = \mathbf{H}' + 4\pi\mathbf{I}' \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (\mathrm{D})\ (7).$$

$$4\pi\mathbf{C}' = \mathrm{V}\nabla'\mathbf{H}' \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (\mathrm{E})\ (8).$$

$$\mathrm{d}' = \mathrm{K}'\mathbf{E}'_0/4\pi \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (\mathrm{F})\ (9).$$

$$\mathbf{K}' = \mathrm{R}'^{-1}\mathbf{E}'_0 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (\mathrm{G})\ (10).$$

Equation (H) we also omit as in this, the present theory certainly gives a result different from MAXWELL's.

$$\mathbf{B}' = \mu'\mathbf{H}' \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (\mathrm{L})\ (11),$$

"when the magnetisation arises from the magnetic induction," MAXWELL adds. The equations omitted are

$$\text{Mechanical force due to field} = \mathrm{V}\mathbf{C}'\mathbf{B}' - \mathrm{D}'\nabla' z - n'\nabla'\Omega \quad . \quad (\mathrm{C})\ (12).$$

$$\mathbf{C}' = \mathbf{K}' + \partial\mathrm{d}'/\partial t \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (\mathrm{H})\ (13).$$

$$\mathrm{D}' = -\mathrm{S}\nabla'\mathrm{d}'^{*} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (\mathrm{J})\ (14).$$

$$n' = \mathrm{S}\nabla'\mathbf{I}' \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (15).$$

* The omission of the *minus* sign in MAXWELL's equation $e = \mathrm{S}\nabla\mathfrak{D}$ is obviously a misprint. [See equation (J) § 612.]

In these, D′, n' have been substituted for MAXWELL's e, m, as the latter symbols already have, in the present paper, a different meaning. "When the magnetic force can be derived from a potential"

$$\mathbf{H}' = -\nabla'\Omega \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (16).$$

[There is no risk of this scalar Ω being confused with the Ω of §§ 9, 10, 54 of the present paper.] In addition to these, he gives in § 613 the surface equation corresponding to equation (14), viz.,

$$D_s' = [SU\nu'\mathrm{d}']_{a+b} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (K)\ (17),$$

where D_s' has been put for his σ.

61. Equation (5) is the same as equation (22), § 26, above, (7) as (23) § 26, (8) as (14) § 25. We can now show that equations (6), (9), (10), (11) all follow if we assume that there is no external force other than that due to friction.

By the last paragraph of § 50 above, we see that what MAXWELL calls **E** is not likely to be what on the present theory we call **E**′. To compare with ordinary theories, then, it is convenient to introduce a new intensity $\mathbf{E}_0$ defined by

$$\mathbf{E}_0 = \mathrm{R}\mathbf{K} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (18).$$

Since $\mathbf{E}_0$ and R**K** (§ 10, Prop. VI, above) are both intensities, equation (10) follows. To prove (9), note that

$$_{c}\nabla x = -\,_{\mathrm{K}}\nabla x = -\mathrm{R}\mathbf{K} = -\mathbf{E}_0$$
$$_{d}\nabla l = -4\pi\mathrm{K}^{-1}\mathrm{d},$$

so that putting e of equation (28), § 50, equal to zero,

$$\mathrm{d} = \mathrm{K}\mathbf{E}_0/4\pi,$$

from which equation (9) follows by Prop. VI, § 10. Again,

$$\mathbf{B} = 4\pi_{\mathrm{H}}\nabla l = \mu\mathbf{H} + 4\pi\mathbf{I}_0$$

and therefore

$$\mathbf{B}' = \mu'\mathbf{H}' + 4\pi\mathbf{I}_0' \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (19),$$

which implies that the part of **B**′ "induced" by magnetic force, is $\mu'\mathbf{H}'$. This is equation (11).

62. To prove equation (6), note first that putting **E** = 0, equation (29) of § 50 [modified by equation (39) of § 50] gives

$$\mathbf{E}_0 = -\dot{\mathbf{A}} - \nabla\psi \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (20).$$

Next note that by d/dt, or a dot, is denoted differentiation with regard to time, which follows the motion of matter, and by $\partial/\partial t$, a differentiation at a fixed point of space, so that d/dt is commutative with ∇, but not with ∇', and $\partial/\partial t$ with ∇', but not with ∇. Hence, as is well known,

$$d/dt = -\,\mathrm{S}\dot{\rho}'\nabla' . + \partial/\partial t \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (21).$$

Now, by equation (20),

$$\mathbf{E}'_0 = \chi'^{-1}\mathbf{E}_0 = -\,\chi'^{-1}d(\chi'\mathbf{A}')/dt - \nabla'v$$
$$= -\,\dot{\mathbf{A}}' - \chi'^{-1}\dot{\chi}'\mathbf{A}' - \nabla'v.$$

Now [former paper, equation (25)],

$$\chi'\omega = -\,\nabla_1\mathrm{S}\omega\rho'_1,$$

so that

$$-\,\chi'^{-1}\dot{\chi}'\mathbf{A}' = \chi'^{-1}\nabla_1\mathrm{S}\mathbf{A}'\dot{\rho}'_1 = \nabla'_1\mathrm{S}\mathbf{A}'\dot{\rho}'_1,$$

and, by equation (21),

$$-\,\dot{\mathbf{A}}' = -\,\partial\mathbf{A}'/\partial t + \mathrm{S}\dot{\rho}'\nabla' . \mathbf{A}',$$

therefore

$$\mathbf{E}'_0 = -\,\partial\mathbf{A}'/\partial t + \mathrm{S}\dot{\rho}'\nabla' . \mathbf{A}' + \nabla'_1\mathrm{S}\mathbf{A}'\dot{\rho}'_1 - \nabla'v$$
$$= -\,\partial\mathbf{A}'/\partial t + \mathrm{S}\dot{\rho}'\nabla' . \mathbf{A}' - \nabla'_1\mathrm{S}\mathbf{A}'_1\dot{\rho}' - \nabla'\,(v - \mathrm{S}\mathbf{A}'\dot{\rho}')$$
$$= -\,\partial\mathbf{A}'/\partial t + \mathrm{V}\dot{\rho}'\mathrm{V}\nabla'\mathbf{A}' - \nabla'z$$
$$= -\,\partial\mathbf{A}'/\partial t + \mathrm{V}\dot{\rho}'\mathbf{B}' - \nabla z,$$

where

$$z = v - \mathrm{S}\mathbf{A}'\dot{\rho}' \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (22).$$

This proves equation (6). Of course, this more complicated form of equation (20) is necessary for some purposes, but the simpler form is more useful in discussing the general theory. From the simpler form, indeed, we may see at once that MAXWELL'S result must follow, since it implies the truth of the principle from which he deduces his result. That principle is ('Elect. and Mag.,' 2nd ed., § 598) that the line integral of $\mathbf{E}'_0$ round any closed curve moving with matter equals the rate of decrease of the line integral of $\mathbf{A}'$ round the same curve. Since both $\mathbf{E}_0$ and $\mathbf{A}$ are intensities, this may in our notation be expressed by saying that the line integral of $\mathbf{E}_0$ round the corresponding *fixed* curve equals the rate of decrease of the line integral of $\mathbf{A}$ round the fixed curve. This last is clearly insured by the equation $\mathbf{E}_0 = -\,\dot{\mathbf{A}} - \nabla v$.

Thus in the results contained in equations (5) to (11) the present theory is in complete agreement with MAXWELL'S. Equations (14), (15), (17) may be taken as definitions. Equations (12), (13), (16) remain. Of these the last implies several other equations involving Ω and **I**. It may be left for the present. *On the present theory equations* (12), (13) *are not true.*

63. It remains then to investigate the physical bearing of the points of difference. Equation (13), of course, could not be expected to represent the results of the present theory, from the definition of a current adopted in § 4 above. Equation (13) asserts that the dielectric current is $\partial \mathbf{d}'/\partial t$. The question is by what on the present theory this statement must be replaced. Since $\mathbf{c}$ a flux $= \dot{\mathbf{d}}$,

$$\mathbf{c}' = m^{-1}\chi\mathbf{c} = m^{-1}\chi d\,(m\chi^{-1}\mathbf{d}')/dt = \dot{\mathbf{d}}' + m^{-1}\chi\frac{d}{dt}(m\chi^{-1})\,\mathbf{d}'.$$

Now by former paper equations (9), (11),

$$m\chi^{-1}\omega = -\tfrac{1}{2}\mathrm{V}\nabla_1\nabla_2\mathrm{S}\omega\rho'_1\rho'_2.$$

Hence

$$\frac{d}{dt}(m\chi^{-1})\,\omega = -\mathrm{V}\nabla_1\nabla_2\mathrm{S}\omega\dot{\rho}'_1\rho'_2 = \mathrm{V}\nabla_1\chi'\mathrm{V}\omega\dot{\rho}'_1\ [\textit{ibid.}, \text{equation } (25)]$$
$$= \mathrm{V}\chi'\nabla'_1\chi'\mathrm{V}\omega\dot{\rho}'_1 = m\chi^{-1}\mathrm{V}\nabla'_1\mathrm{V}\omega\dot{\rho}'_1,$$

by TAIT's 'Quaternions,' 3rd. ed., § 157, equation (2). Hence

$$\left.\begin{aligned}\mathbf{c}' &= \dot{\mathbf{d}}' + \mathrm{V}\nabla'_1\mathrm{V}\mathbf{d}'\dot{\rho}'_1 = \partial\mathbf{d}'/\partial t + \mathrm{V}\nabla'\mathrm{V}\mathbf{d}'\dot{\rho}' - \dot{\rho}'\mathrm{S}\nabla'\mathbf{d}' \\ \mathbf{C}' &= \dot{\mathbf{D}}' + \mathrm{V}\nabla'_1\mathrm{V}\mathbf{D}'\dot{\rho}'_1 = \partial\mathbf{D}'/\partial t + \mathrm{V}\nabla'\mathrm{V}\mathbf{D}'\dot{\rho}'\end{aligned}\right\} \quad . \quad . \quad . \quad (23),$$

which equations have already been given in anticipation, in equation (38), § 50. In the case of an incompressible substance (solid or fluid) $\mathrm{S}\nabla'\dot{\rho}' = 0$, and, therefore,

$$\mathbf{c}' = \dot{\mathbf{d}}' + \mathrm{S}\mathbf{d}'\nabla'\,.\,\dot{\rho}' \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (24),$$

and for a rigid body whose angular velocity (vector) is η this simplifies further to

$$\mathbf{c}' = \dot{\mathbf{d}}' - \mathrm{V}\eta\mathbf{d}' \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (25).$$

Thus, on the present theory neither $\dot{\mathbf{d}}'$ nor $\partial\mathbf{d}'/\partial t$ is the dielectric current.

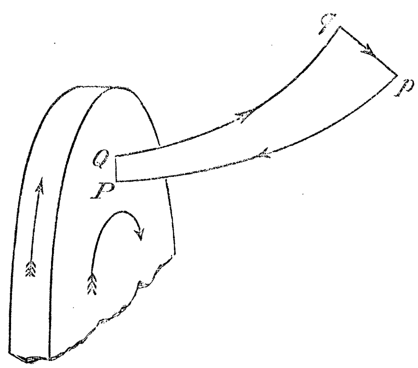
The effect of the difference between the theories will be very slight in most experimental work, though it will, of course, lead to different results in the solution of certain problems which involve currents in moving bodies.

64. There is one experimental result, however, in connection with which equation (23) has considerable interest. In the 'Phil. Mag.,' V., vol. xxvii [1889], p. 445, Professor

ROWLAND and Mr. HUTCHINSON describe the experiments by which they have proved that a moving charged body acts on surrounding bodies as it should on the convection current theory. Now, this can be shown *accurately* to follow from equation (23) if we make the double assumption (1) that the medium in contact with the moving conductor is at rest, and (2) that the slipping which thus takes place may be regarded as the limit of a rapid shear; and *approximately* to follow without the assumption.

First, then, assume there is no slipping. Let the motion be steady. If the moving dielectric be itself charged, we see by the term $-\dot{\rho}'\mathrm{S}\nabla'\mathbf{d}'$ in $\mathbf{c}'$ that the effect of its motion is to cause the current due on the convection current theory to its charge. Since the motion is steady, $\partial\mathbf{d}'/\partial t = 0$. To take account of the remaining term $\mathrm{V}\nabla'\mathrm{V}\mathbf{d}'\dot{\rho}'$ of $\mathbf{c}'$, consider the current through a strip of surface constructed thus:—Take an elementary line PQ in the surface of the conductor. Through all points of PQ draw the lines of electrostatic induction (lines at every point of which the tangent is parallel to $\mathbf{d}'$). Bound the strip of surface thus obtained at any distance from PQ by another element *pq*. In fig. 1 the arrows indicate (1) the direction of motion of the conductor, (2) the

Fig. 1.



positive direction (PQ*qp*) *round* the strip when the positive direction *through* it is that of the motion at PQ. The current through PQ*qp* $= -\iint \mathrm{S}\mathbf{c}'d\Sigma'$ taken over the strip. The part contributed to this by the term $\mathrm{V}\nabla'\mathrm{V}\mathbf{d}'\dot{\rho}'$ of $\mathbf{c}'$ is

$$-\iint \mathrm{S}d\,\Sigma'\nabla'\mathrm{V}\mathbf{d}'\dot{\rho}' = -\int \mathrm{S}d\,\rho'\mathbf{d}'\dot{\rho}'$$

by equation (3), § 5. The parts contributed to the line integral by the lines of induction *p*P, Q*q* are zero. Hence the current through the strip

$$= -\mathrm{S}\overline{\mathrm{PQ}}\mathbf{d}'\dot{\rho}' + \mathrm{S}\overline{pq}\mathbf{d}'\dot{\rho}'$$

where $\overline{\mathrm{PQ}}$, $\overline{pq}$ stand as usual for the *vectors* PQ, *pq*. The first of these terms is the rate of flow in the direction of motion of electrostatic charge through the element

PQ. Hence, if at pq $\dot{\rho}'$ is small enough to be neglected in the above expression, the whole current which on the present theory would be flowing through the strip PQqp is the same as the current due to surface charge, which on the convection current theory would be flowing in the same direction across the element PQ. The extension to the case when slipping is allowed and the dielectric is at rest is obvious.

With regard to the plausibility of this explanation, it must be remembered that in this paper we admittedly do not take account of the independent motion in the very same space of two mediums such as air and ether. Now, probably,* the ether is at rest relative to the conductor, and it is reasonable to suppose that the relative motion of the conductor and the ether is of more importance in connection with the part $\mathrm{V}\nabla'\mathrm{V}\,\mathbf{d}'\dot{\rho}'$ of $\mathbf{c}'$ than the relative motion of the air and the conductor. On the other hand, as the air carries about with it any charge it possesses, it is the motion of the air we must consider in interpreting the term $-\dot{\rho}'\mathrm{S}\nabla'\mathbf{d}'$. Indeed, if we suppose the ether only to *bound* the conductor and the molecules of air, and that the ether is mainly at rest (*i.e.*, acts to the conductor and the molecules of air much as an ocean of perfect fluid, which could *slide* over surfaces, and was originally at rest, would act to the conductor and molecules supposed immersed in it) the explanation is *complete.* [I do not wish to imply that I endorse this theory of the relative behaviour of the ether and matter].

On the whole, I think it may be said that this test of the correctness of eq. (23) is fairly well met.

65. Before comparing eq. (12) with the corresponding results of the present theory, it is necessary to make one or two remarks on passages from MAXWELL'S 'Electricity and Magnetism.' In the quotations I am about to make I have in every case changed MAXWELL'S notation to the notation used above, as leading to a clearer comparison of results. Consistently with this, I have always substituted Quaternion language for the corresponding Cartesian.

In the first place† I wish to discuss MAXWELL'S views concerning the scalar he calls Ψ, and which has been above denoted by z [equations (6) (12) (22)]. In his second volume he seems to intend the symbol always to have the same meaning. The first place in which it occurs in this volume is in § 598, where he is investigating the expression for $\mathbf{E}'_0$. After proving that

$$\mathbf{E}'_0 = \mathrm{V}\dot{\rho}'\mathbf{B}' - \partial\mathbf{A}'/\partial t - \nabla'z,$$

he proceeds: "The terms involving the new quantity z are introduced for the sake of giving generality to the expression for $\mathbf{E}'_0$. They disappear from the integral when

* According to a report in 'Nature,' September, 1891, p. 454, Professor LODGE described to the British Association experiments which go to prove this. I have not yet seen details of the experiments.

† Before going further, attention may be recalled to the footnote of § 38 above.

extended round the closed circuit. The quantity z is, therefore, indeterminate as far as regards the problem now before us, in which the total electromotive force round the circuit is to be determined. We shall find, however, that when we know all the circumstances of the problem, we can assign a definite value to z, and that it represents, according to a certain definition, the *electric potential* at the point ρ'." Now, I have looked in vain through the subsequent part of his treatise to find the promised definition of electric potential, and I have tried hard on MAXWELL'S own assumptions to see how the *definite value* he here speaks of is to be assigned, and I have totally failed. He nowhere shows how to assign a definite value to $\mathbf{A}'$; whereas he certainly assigns a definite value to $\mathbf{B}'$, and also from equations (9) (10) above, he also clearly assigns a definite value to $\mathbf{E}_0'$. From the equation just given, then, it follows that $\nabla' z$ must be indefinite in order to counterbalance the arbitrary part of $\partial\mathbf{A}'/\partial t$, which is necessarily of the form ∇' (*some scalar*).* Leaping over this difficulty of MAXWELL'S assertions, however, *i.e.*, supposing $\partial\mathbf{A}'/\partial t$ definite, the question still remains what is the definite value of z? Light *seems* to be thrown on the question by the assertion above that it is the "electric potential," and the following, taken from § 630 of his treatise :—

"The energy of the system may be divided into the Potential Energy and the Kinetic Energy.

"The potential energy, due to electrification, has already been considered in § 85. It may be written

$$\mathrm{W} = \tfrac{1}{2}\Sigma\mathfrak{D}z,$$

where $\mathfrak{D}$ is the charge of electricity at a place where the electric potential is z, and the summation is to be extended to every place where there is electrification.

"If $\mathbf{d}'$ is the electric displacement, the quantity of electricity in the element of volume $d\varsigma'$ is

$$\mathfrak{D} = -\,\mathrm{S}\nabla'\mathbf{d}'d\varsigma',$$

and

$$\mathrm{W} = -\,\tfrac{1}{2}\iiint z\mathrm{S}\nabla'\mathbf{d}'d\varsigma'$$

where the integration is to be extended throughout all space." He then shows that it follows that

$$\mathrm{W} = \tfrac{1}{2}\iiint \mathrm{S}\mathbf{d}'\nabla' z d\varsigma',$$

and proceeds :

* For $\mathrm{V}\nabla'\,\mathbf{A}'$ is assigned for every point of space and $[\mathrm{V}d\Sigma'\mathbf{A}']_{a+b} = 0$. It is well known that when this much and no more of a vector is known, it contains an arbitrary term ∇' (*a scalar*), and that this is the full extent of its arbitrariness.

"If we now write $\mathbf{E}_0'$ the electromotive force instead of $-\nabla' z$, we find

$$W = -\tfrac{1}{2}\iiint S\mathbf{d}'\mathbf{E}_0' d\varsigma'.$$

"Hence, the electrostatic energy of the whole field will be the same if we suppose that it resides in every part of the field where electrical force and electrical displacement occur, instead of being confined to the places where free electricity is found." Were it not for this last statement, the interpretation I should put on the whole of the above passage would be expressed thus:—In the *particular case* of electrostatics $\mathbf{E}_0' = -\nabla' z$ and $W = -\frac{1}{2}\iiint S\mathbf{d}'\mathbf{E}_0' d\varsigma'$. In the *general case*, where the electricity is not stationary, $\mathbf{E}_0'$ cannot be put in the form $-\nabla' z$; but *we shall nevertheless assume* that the equation $2W = -\iiint S\mathbf{d}'\mathbf{E}_0' d\varsigma'$ is still true. This seems to me the interpretation that presents least difficulty, but it seems hard to reconcile it with the last sentence quoted, which implies that the equation $2W = \iiint S\mathbf{d}'\nabla' z d\varsigma'$ is *exactly the same* as the equation $2W = -\iiint S\mathbf{d}'\mathbf{E}_0' d\varsigma'$. There seems only one other possible interpretation of the passage, but that lands us in hopeless difficulties. This explanation is that the $\mathbf{E}_0'$ which occurs in § 598, where it *cannot be put* in the form $-\nabla' z$, has a different meaning from the $\mathbf{E}_0'$ which occurs in §§ 630, 631, where it is $= -\nabla' z$. If he has changed the meaning of $\mathbf{E}_0'$, we may presume that matters have not been further complicated by a change in the meaning of z. In this case §§ 630, 631 may be put thus:—

(1) It is assumed that the energy of the field can be divided into two parts, electrostatic and electromagnetic.

(2) The former of these, in the absence of electric currents, can be put in the form $\frac{1}{2}\iiint S\mathbf{d}'\nabla' z d\varsigma'$ where z is a scalar. It is assumed that this statement is also true when there are electric currents present.

(3) It is assumed that the z appearing in this expression is the same as the z which occurs in the general equation $\mathbf{E}_0' = V\dot{\rho}'\mathbf{B}' - \partial\mathbf{A}'/\partial t - \nabla' z$; and it is convenient to give it the name electric potential.

It will be acknowledged that these assumptions are more unwarrantable than the one required for the first interpretation, and therefore I shall understand the passage to be thus, as at first, correctly interpreted. But if this be so, we are as far off as ever from the conclusion that z has a definite value which can appropriately be called the electric potential.

66. This is no mere question of terms, for [equation (12), above] MAXWELL asserts that in the expression for the force due to the field occurs a term $-D'\nabla' z$, and here the indefiniteness is not counterbalanced by the corresponding indefiniteness of $\partial\mathbf{A}'/\partial t$. There are more ways than one of compromising to get out of the difficulty.* The

* For instance, we may (arbitrarily) render $\mathbf{A}'$ definite by the equations $S\nabla'\mathbf{A}' = 0\ [Sd\Sigma'\mathbf{A}']_{a+b} = 0$, and thus render $\nabla' z$ definite; and we may then *assert* that equation (12) is correct.

course followed here is, of course, to abide by what the present theory leads to, and then to choose that particular interpretation of the above passages which appears least at variance with our results. [It will be seen from the above that the statements at the end of § 50, above, are true, viz., that the potential v of the present theory is certainly not [equation (22), § 62] the same as MAXWELL's potential z, and that without some such assumption as $S\nabla'\mathbf{A}' = 0$, $[Sd\Sigma'\mathbf{A}']_{a+b} = 0$ our potential, like MAXWELL's, is indefinite apart from an arbitrary additive constant. *This* question of the arbitrariness of the potential *is* one merely of terms.]

It is, perhaps, unnecessary now to say that the position I wish to maintain is that MAXWELL has not investigated in a perfectly general manner the consequences of his own theory, and that, consequently, some of his *general* equations may prove inconsistent with that theory. Equation (12) I hold to be such an equation. So little right, indeed, has he to put this down as one of his general results that it is, I hold, inconsistent with other parts of his treatise. For instance, if the equation were consistent with equation (4), § 640 ('Elect. and Mag.,' 2nd edition), we should have $\nabla_1'S\mathbf{H}_1'\mathbf{I}' = \nabla'\Omega S\nabla'\mathbf{I}'$, which is certainly* not the case in general on MAXWELL's theory. I shall not, then, compare the mechanical results of the present theory with equation (12) at all, but shall adopt the simpler process of comparing the stress which results from the present theory with that which MAXWELL obtains in Chapter V. of Part I., and Chapter XI. of Part IV.

67. Before this comparison another matter must be considered. MAXWELL, in accordance with, I think, universal custom, supposes that a molecular couple exists due to magnetism. In the first place this extraordinary exception to our ordinary

* As might be expected, the relation is true in very many important problems whose details have been worked out, but it is not true in general, even when there are no currents. Dropping the special notation of this paper for the moment, let r, x have their usual Cartesian meanings. Denote differentiations with regard to r by dashes. Let F be any function of r. If there be no currents, and if

$$\Omega = x\mathrm{F},$$

then will

$$-\mathbf{H} = \nabla\Omega = i\mathrm{F} + \rho x\mathrm{F}'/r,$$

and [from the relation $S\nabla(\mathbf{H} + 4\pi\mathbf{I}) = 0$, which is the only equation to be satisfied]

$$\mathbf{I} = i(r\mathrm{F}' + 3\mathrm{F})/4\pi.$$

In this case

$$4\pi\nabla_1 S\mathbf{I}\mathbf{H}_1 = -4\pi S\mathbf{I}\nabla.\nabla\Omega = (r\mathrm{F}' + 3\mathrm{F})\{i2x\mathrm{F}'/r + \rho[\mathrm{F}' + x^2 d(\mathrm{F}'/r)/dr]/r\},$$

and

$$4\pi\nabla\Omega S\nabla\mathbf{I} = -(x/r)\,d(r\mathrm{F}' + 3\mathrm{F})/dr.\{i\mathrm{F} + \rho x\mathrm{F}'/r\},$$

which are clearly not in general equal. The above expression for $\mathbf{I}$ is, of course, not the general one for this case, as we may add to it a term $V\nabla\sigma$ where σ is any vector. Also, it is assumed that F is such that both $\mathbf{H}$ and $\mathbf{I}$ are everywhere continuous, *i.e.*, F and F′ are everywhere continuous. For instance, put $\mathrm{F} = (r - a)^2$ from $r = 0$ to $r = a$, and $\mathrm{F} = 0$ from $r = a$ to $r = \infty$.

conceptions of stress seems to me quite unnecessary on general grounds. It is well known that to every magnetic distribution there is an analogous conceivable distribution of ordinary statical electricity. In the ordinary action-at-a-distance theories the mutual mechanical effects of different parts of a magnetic system would be exactly the same as the corresponding effects in the analogue. Why, then, should it be considered unnecessary in the case of electrostatics, but necessary in the case of magnetics, to postulate a molecular couple? Why not, in other words, say that the stress which MAXWELL would suppose existent in the electric analogue is exactly the stress really existent in the magnetic system? In the second place, although the process seems viciously needless, we may, if we like, conceive any physical phenomena involving stress as causing a molecular couple which is exactly balanced by a stress-couple. [It must be so equilibrated in order to insure against infinite angular acceleration of an element of matter—supposing, of course, that the ultimate constitution of matter were not heterogeneous.] This latter stress-couple will be entirely of the nature of a reaction, since (former paper, p. 108) it is entirely independent of the potential energy of strain. In the present case, then, in which we suppose electromagnetic phenomena to produce stress, we shall have one stress exactly equilibrating another stress, neither of them having anything to do with the Lagrangian function. This is only another way of saying that no physical conception whatever is gained by the supposition that the particular physical phenomenon produces a stress-couple. We shall, then, consider it necessary to compare our results only with the pure part of the stress which MAXWELL supposes to exist.

Thus in § 641 MAXWELL arrives at the conclusion that the stress required to produce observed electromagnetic phenomena is v where

$$8\pi v\omega = -2\mathbf{H}'S\omega\mathbf{B}' + \omega\mathbf{H}'^2 = 8\pi\{\phi'\}\omega + V(V\mathbf{B}'\mathbf{H}'.\omega) \quad . \quad . \quad . \quad (26),$$

where $\{\phi'\}$ denotes the pure stress given by

$$8\pi\{\phi'\}\omega = -\mathbf{H}'S\omega\mathbf{B}' - \mathbf{B}'S\omega\mathbf{H}' + \omega\mathbf{H}'^2 = -V\mathbf{B}'\omega\mathbf{H}' - 4\pi\omega S\mathbf{I}'\mathbf{H}' \quad . \quad (27).$$

Now, what in the former paper (p. 108) was called the couple stress part of v, namely, $V(V\mathbf{B}'\mathbf{H}'.\omega)/8\pi$, produces a couple per unit volume $V\mathbf{B}'\mathbf{H}'/4\pi = V\mathbf{I}'\mathbf{H}'$, and this must be equilibrated by some other couple per unit volume even when the body is not in equilibrium. This couple can only result from a couple stress, $V\epsilon\omega$, which produces a couple, 2ϵ, per unit volume; and this is quite independent of the potential energy of strain, and therefore of the Lagrangian function. Thus, $2\epsilon + V\mathbf{I}'\mathbf{H}' = 0$. If now, in addition to the stress $v\omega$ we take account of the *reactionary* stress $V\epsilon\omega$, we simply get the pure stress $\{\phi'\}\omega$. We shall then merely compare the stress (pure) which flows from the present theory with $\{\phi'\}$.

68. It is well-known that MAXWELL's stress can only be looked on as a normal

type of stress. It by no means explains all the known facts. It does explain satisfactorily such known mechanical actions of real conductors—conveying currents and bearing charges—and magnets on one another as are of the nature of apparent actions at a distance. It does not at all explain the many known mechanical actions of one part of a conductor or magnet on another part which can be tested only by observing the (small) strains resulting. In other words, conductors and magnets are found to behave mechanically, as they would if MAXWELL'S supposed stress acted outside them, but not as if this stress existed internally. It need not therefore be matter for surprise if, on the present theory, what would appear as the most suitable stress to regard as normal should differ from MAXWELL'S. It is only necessary that *just outside* conductors and magnets it should be identical with MAXWELL'S.

To see what on the present theory should be regarded as a normal type we must discuss, from the physical point of view, the results of §§ 54, 55 above. As far as I can see (but this is, of course, largely a matter for personal judgment) on the present theory we should recognize two normal types of stress—one for fluids and one for solids. The reason is that we may assume fluids to be magnetically and electrically isotropic, and that fluids are subject to indefinitely large strains. On the other hand, solids, even if magnetically and electrically isotropic when unstrained, cannot be considered so when strained and, moreover, their strains cannot exceed a certain—usually very small—amount without the form of l being permanently altered.

For bodies which are electrically isotropic, however large their strain, it is needless to say that we must regard the Lagrangian function as given in terms of the dashed letters. For such bodies, η of equation (19) § 54 is zero. [For let α'', β'' . . . be the vectors of which l'' is an explicit function. Since the body is isotropic the value of l'' must remain unaltered if we rotate α'', β'' . . . all to the same extent round the same axis. In particular, if we increase α'', β'' . . . by $V\epsilon\alpha''$, $V\epsilon\beta''$, . . . where ϵ is an infinitely small vector, l'' must remain unaltered, *i.e.*, the increment $-\Sigma S\epsilon\alpha''_{\alpha}\nabla''l'' = 0$. Since ϵ is arbitrary, it follows that $\Sigma V\alpha''_{\alpha}\nabla''l'' = 0$]. By equation (20) § 54 above, we see that the assumption that $[\phi'] = 0$ amounts to assuming that *the Lagrangian function of unit volume of the body when strained, however largely, is the same as the Lagrangian function of unit volume of the body when unstrained.* By equation (29), § 55, we see that the assumption $\phi_0' = 0$ amounts to assuming that *that part of the Lagrangian function which causes the body to differ from vacuum, and which is contributed by a given mass of the body, is unaffected by strain.* Hence from equations (21), § 54, and (30), § 55, we have

For an isotropic body, of which the Lagrangian function per unit volume is unaffected by strain

$$\left.\begin{aligned} 2\phi' = -\{2l' + \Sigma S(\tau'_{\tau}\nabla'l' + \sigma'_{\sigma}\nabla'l')\} \\ + \Sigma V\{\tau'(\ )_{\tau}\nabla'l' - \sigma'(\ )_{\sigma}\nabla'l'\} \end{aligned}\right\} \quad . \quad (28).$$

For an isotropic body, of which that part of the Lagrangian function per unit mass, which causes the body to differ from vacuum, is unaffected by strain

$$\left.\begin{aligned} 2\phi' = (\mu_0 \mathbf{H}'^2/4\pi - 4\pi \mathrm{K}_0^{-1}\, \mathrm{d}'^2) - \Sigma \mathrm{S}\, (\tau'_\tau \nabla' l' + \sigma'_\sigma \nabla' l') \\ + \Sigma \mathrm{V} \{\tau' (\ )_\tau \nabla' l' - \sigma' (\ )_\sigma \nabla' l'\} \end{aligned}\right\} \quad . \quad (29).$$

It is scarcely necessary to say, that of course it is not meant to be here implied that there is any body whatsoever whose general Lagrangian function—whether per unit mass or per unit volume—is even approximately unaffected by strain. It is only for brevity that we verbally contemplate such a body. There seems little or no reason for choosing one rather than the other of these two stresses as the normal type of stress for fluids. Both of them would satisfy the condition that for a gas which, however large its strain, always behaved like a vacuum, the normal stress would be the vacuum stress which resulted from identical values of $\mathbf{H}'$ and d'. As the stress of equation (29), however, agrees more closely with the pure part of MAXWELL's stress than that of equation (28), we will call the stress of (29) the normal stress for fluids.

For solids it is harder to find a suitable normal stress, but as by far the greater number of them (non-magnetic bodies) behave magnetically approximately like a vacuum, it seems to me that the most suitable is obtained by supposing $\text{Ꞁ}l_0$ of equation (27), § 55 to be zero. In this case, of course, we assume that for solids the normal type of stress is the stress that, with identical values of $\mathbf{H}'$ and d' would exist in a vacuum. Thus

$$\textit{For a vacuum, } \phi'\omega = -2\pi \mathrm{K}_0^{-1}\, \mathrm{d}'\omega \mathrm{d}' - \mu_0 \mathbf{H}'\omega \mathbf{H}'/8\pi \quad . \quad . \quad . \quad (30),$$

but it is needless to say that this is not perfectly satisfactory. The question may be asked why, in the present case, the stress of equation (29) should not be still retained as the normal one? The answer is, that the equation [(30), § 55] from which it is derived, and which actually must, in every exact discussion, be taken in its place, is a wholly unsuitable one for a solid, while equation (27), § 55, is a suitable one. Again it may be asked, Why not retain the stress equation in its original form $\phi' = -2m^{-1} \chi \text{Ꞁ} l \chi'$, for solids? The answer to this is, that the important fact that the great majority of solids behave magnetically like a vacuum is not thereby readily taken account of.

69. To compare these stresses and their effects with MAXWELL's, it must first be noted that MAXWELL has only investigated the electrostatic part of his stress for the case of a series of charged conductors surrounded by a dielectric that behaves electrostatically like a vacuum. I consider myself at liberty then to substitute anything for the electrostatic part of his stress which reduces to his for that particular case. The stress he obtains in Chapter V. of Part I. of his treatise is $-2\pi \mathrm{K}_0^{-1}\, \mathrm{d}' (\ ) \mathrm{d}'$. For the particular case mentioned this may be written

$$\mathrm{Vd}'(\ )\,{}_{\mathrm{d}}\nabla' l'/2 - \mathrm{Sd}'({}_{\mathrm{d}}\nabla' l' + 4\pi \mathrm{K}_0^{-1}\,\mathrm{d}')/2,$$

since for that case ${}_{\mathrm{d}}\nabla' l' = -4\pi \mathrm{K}_0^{-1}\,\mathrm{d}'$. We shall assume that this is the correct expression in general, since thereby the stress of equation (29) is rendered identical with the pure part of MAXWELL's stress. The pure part of his electromagnetic stress is the $\{\phi'\}$ of equation (27) above. Let us then put

$$\phi_m'\omega = \mathrm{Vd}'\omega_{\mathrm{d}}\nabla' l'/2 - \omega \mathrm{Sd}'({}_{\mathrm{d}}\nabla' l' + 4\pi \mathrm{K}_0^{-1}\,\mathrm{d}')/2 - \mathrm{V}\mathbf{B}'\omega\mathbf{H}'/8\pi - \omega \mathrm{S}\mathbf{I}'\mathbf{H}'/2 \quad (31),$$

or, if we assume that the *complete* expression for x' is $-\mathrm{S}\mathbf{K}'\mathrm{R}'\mathbf{K}'/2$ [equations (28), § 50 and (20), § 35],

$$\phi_m'\omega = -\mathrm{Vd}'\omega\mathbf{E}'_0/2 + \omega\mathrm{Sd}'(\mathbf{E}'_0 - 4\pi\mathrm{K}_0^{-1}\,\mathrm{d}')/2 - \mathrm{V}\mathbf{B}'\omega\mathbf{H}'/8\pi - \omega\mathrm{S}\mathbf{I}'\mathbf{H}'/2 \quad (31a),$$

and call ϕ_m' MAXWELL's stress. [Of course I do not thereby mean to render MAXWELL responsible for this form.] If we regard $\mathrm{Vd}'(\ )\,{}_{\mathrm{d}}\nabla' l'/2$ as the correct generalisation of MAXWELL's electrostatic stress we may indicate it by calling $[\phi_m']$ the second form of MAXWELL's stress where

$$[\phi_m']\,\omega = \mathrm{Vd}'\omega_{\mathrm{d}}\nabla' l'/2 - \mathrm{V}\mathbf{B}'\omega\mathbf{H}'/8\pi - \omega\mathrm{S}\mathbf{I}'\mathbf{H}'/2 \quad . \quad . \quad . \quad . \quad (32),$$

which gives, on the assumption that the complete expression for x' is $-\mathrm{S}\mathbf{K}'\mathrm{R}\,\mathbf{K}'/2$,

$$[\phi_m']\,\omega = -\mathrm{Vd}'\omega\mathbf{E}'_0/2 - \mathrm{V}\mathbf{B}'\omega\mathbf{H}'/8\pi - \omega\mathrm{S}\mathbf{I}'\mathbf{H}'/2 \quad . \quad . \quad . \quad (32a).$$

If we now assume that the only variables of l' are $\mathbf{H}'$ and d', equation (28) gives

$$2\phi' = -\{2l' + \mathrm{S}(\mathrm{d}'_{\mathrm{d}}\nabla' l' + \mathbf{B}'\mathbf{H}'/4\pi)\} + \mathrm{Vd}'(\quad)\,{}_{\mathrm{d}}\nabla' l' - \mathrm{V}\mathbf{B}'(\quad)\,\mathbf{H}'/4\pi \ . \ (28a),$$

of which the following particular cases should be noted :—

If l' be quadratic in d' *and* $\mathbf{H}'$,
$$\left.\phi' = \mathrm{Vd}'(\quad)\,{}_{\mathrm{d}}\nabla' l'/2 - \mathrm{V}\mathbf{B}'(\quad)\,\mathbf{H}'/8\pi\right\} \quad . \quad . \quad . \quad . \quad . \quad . \quad (28b),$$

If l' be given by (3), § 59, *and x' by* (4),
$$\left.\phi' = -\mathrm{Vd}'(\quad)\,\mathbf{E}'_0/2 - \mathrm{V}\mathbf{B}'(\quad)\,\mathbf{H}'/8\pi + \mathrm{S}\mathbf{I}_0'\mathbf{H}'/2\right\} \quad . \quad . \quad . \quad (28c),$$

from which it follows from equation (32*a*) that in this case

$$\phi' = [\phi_m'] + \mathrm{S}(\mathbf{I}'_0 + \mathbf{I}')\,\mathbf{H}'/2 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (28d,\ 32b),$$

so that this stress differs from the second form of MAXWELL's stress by a hydrostatic pressure which is zero for non-magnetic bodies.

Under the same circumstances (l' a function of $\mathbf{H}'$ and d' only), equation (29) gives

$$\begin{aligned}\phi' = \mathrm{Vd}'(\ )\,{}_{\mathrm{d}}\nabla' l'/2 - \mathrm{Sd}'({}_{\mathrm{d}}\nabla' l' + 4\pi \mathrm{K}_0^{-1}\,\mathrm{d}')/2 \\ - \mathrm{VB}'(\ )\mathbf{H}'/8\pi - \mathrm{SI}'\mathbf{H}'/2 + (\mu_0 - 1)\,\mathbf{H}'^2/8\pi \quad . \quad . \quad . \quad . \quad (29a).\end{aligned}$$

Hence, with the electromagnetic system of units for which $\mu_0 = 1$,

$$\phi' = \phi_m' \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (29b,\ 31b),$$

in which it should be noticed there is no necessity to assume that the complete form of x' is $-\mathrm{SK}'\mathrm{R}'\mathrm{K}'/2$, nor is it assumed, as in equation ($28d$, $32b$) that l' has the particular form given in equation (3), § 59.

To sum up, of the two equations (28) and (29), (28) agrees more closely with MAXWELL as to the electrostatic part, and (29) more closely as to the electromagnetic part. On the whole, equation (29) agrees more closely than (28).*

Of course, the normal stress [eq. (30)] we have adopted for solids is by no means the same as MAXWELL'S, except for non-magnetic bodies whose specific inductive capacities are the same as for a vacuum. But this does not prevent our normal stress explaining all that MAXWELL'S stress explains, and, indeed, from the remarks at the beginning of last section, it is now evident that for all useful purposes either the one stress or the other will serve equally well.

70. We have now compared the results of the present theory with all MAXWELL'S results contained in equations (5) to (17), § 60, above, except (16). Except for equations (12), (13), the agreement is exact, and I think it may now be claimed that what the present theory gives instead of equation (12), agrees, as well as (12), with known facts, and what it gives instead of (13) agrees better than (13).

Equation (16) itself is obvious enough since it merely asserts that $\mathbf{H}'$ has a potential when there are no currents in the field. But it suggests another question—does the present theory lead to the *ordinary* mathematical theory of electromagnetism? It can be easily shown to do so. The mechanical results when expressed in terms of $\mathbf{H}'$ and $\mathbf{I}'$ have just been shown to result in the same forces and moments on conductors and magnets regarded as wholes, as does MAXWELL'S stress. These are all the mechanical demands of the ordinary theory. Equation (19), § 61, shows that the relations between the whole magnetic moment per unit volume, the permanent magnetic moment per unit volume, and the magnetic force at the point, may on the present theory, be regarded as the same as in the ordinary theory. Only one other

* Notwithstanding this, and the fact that I have in this paper called the stress of equation (29) the normal stress, I think equation (28) is to be preferred, partly because of the greater simplicity of the assumptions which lead to it, and partly because of the greater simplicity of the electrostatical results flowing from it.

demand is made by the ordinary theory. $\mathbf{H}'$ and $\mathbf{A}'$ on the one hand, must be determined in terms of $\mathbf{C}'$ and $\mathbf{I}'$, on the other, by means of particular relations.

Let us suppose—merely to get rid of the dashes—the standard position to coincide with the actual position. One difference between MAXWELL's theory and the ordinary theory is that according to the latter it is assumed that each individual magnetic molecule and each elementary current has its own influence—independently of the rest—in producing terms in $\mathbf{A}$ and $\mathbf{H}$. Thus, $\mathbf{H}$ consists of two parts, the first depending only on the magnetism and the second only on the currents. The first $= -\nabla\Omega$, where $\Omega = -\iiint S\mathbf{I}\nabla u d\varsigma$ [MAXWELL's 'Elect. and Mag.,' 2nd ed., § 383, equation (3)], where u^{-1} is the distance of the element $d\varsigma$ from the point under consideration, and where in the differentiations of ∇u the end of u^{-1} at the element $d\varsigma$ is supposed varied. The second part is obtained on the assumption that each closed current causes a term in $\mathbf{H}$ which the corresponding magnetic shell would cause. The second part is thus found to be $\nabla\iiint u\mathbf{C}d\varsigma$.* The first part of $\mathbf{A}$ is supposed to depend on $\mathbf{I}$ in the same way as the $\mathbf{A}$, called the vector potential, of Part III. of MAXWELL's treatise depends, *i.e.*, $= \iiint V\mathbf{I}\nabla u d\varsigma$ [§ 405, equation (22)]. The second part, as with $\mathbf{H}$ is obtained by assuming that any closed current will cause a term in $\mathbf{A}$ equal to the term in $\mathbf{A}$ that would be caused by the corresponding magnetic shell. The second part is thus found to be $\iiint u\mathbf{C}d\varsigma$.† We will suppose that the ordinary theory also admits that $\mathbf{A}$ is arbitrary in containing a term ∇w, where w is a scalar. (This is only to render the comparison with the present theory simpler. Perhaps it ought to be said that the $\mathbf{A}$ thus obtained in terms of $\mathbf{I}$ and $\mathbf{C}$ on the ordinary theory is found to satisfy the conditions $S\nabla\mathbf{A} = 0$, $[Sd\Sigma\mathbf{A}]_{a+b} = 0$, and that the present theory only agrees with the ordinary theory if we arbitrarily impose those relations.) All this may be expressed thus. Defining $\mathbf{A}_0$ and Ω by

$$\mathbf{A}_0 = \iiint u\mathbf{C}d\varsigma,\ \Omega = -\iiint S\mathbf{I}\nabla u d\varsigma \quad \ldots \ldots \quad (33),$$

we shall have

$$\mathbf{A} = \mathbf{A}_0 + \iiint V\mathbf{I}\nabla u d\varsigma + \nabla w \quad \ldots \ldots \quad (34)$$

$$\mathbf{H} = -\nabla\Omega + \nabla\mathbf{A}_0 \quad \ldots \ldots \quad (35).$$

If q be any quaternion function of the position of a point which may be discontinuous at certain surfaces, we have

* The magnetic force at an external point due to a shell of strength $c = c\nabla\iint Sd\Sigma\nabla u = -c\iint Sd\Sigma\nabla.\nabla u$ $= c\iint Vd\Sigma\nabla.\nabla u$ [since $\nabla^2 u = 0$] $= c\int d\rho\nabla u = c\nabla\int u d\rho$. The reason for the change of sign in ∇ on crossing the integral sign is that when outside one end and when inside the other end of u^{-1} is naturally supposed in the differentiations of ∇u to vary.

† This is not inconsistent with §§ 616, 617 of 'Elect. and Mag.,' 2nd edit., for there MAXWELL is considering the two parts together.

$$4\pi q = \nabla^2 \iiint uq d\varsigma = -\nabla \iiint \nabla uq d\varsigma = \nabla \left(\iiint u\nabla q d\varsigma - \iint u d\Sigma q \right).$$

Now, on the present theory (by means of the equations $4\pi\mathbf{C} = \mathrm{V}\nabla\mathbf{H}$, $[\mathrm{V}d\Sigma\mathbf{H}]_{a+b} = 0$, and by elimination of $\mathbf{B}$ from the equations $\mathbf{B} = \mathbf{H} + 4\pi\mathbf{I}$, $\mathrm{S}\nabla\mathbf{B} = 0$, $[\mathrm{S}d\Sigma\mathbf{B}]_{a+b} = 0$),

$$\nabla\mathbf{H} = 4\pi(\mathbf{C} - \mathrm{S}\nabla\mathbf{I}),\ [d\Sigma\mathbf{H}]_{a+b} = -4\pi[\mathrm{S}d\Sigma\mathbf{I}]_{a+b}.$$

Hence substituting $\mathbf{H}$ for q,

$$\mathbf{H} = \nabla\left(\iiint u\mathbf{C}\, d\varsigma - \iiint u\mathrm{S}\nabla\mathbf{I}\, d\varsigma + \iint u\mathrm{S}\, d\Sigma\mathbf{I}\right) = \nabla\left(\iiint u\mathbf{C}\, d\varsigma + \iiint \mathrm{S}\mathbf{I}\nabla u\, d\varsigma\right).$$

This is equation (35). Again substituting $\mathbf{A}$ for q, and putting

$$4\pi w = \iiint u\mathrm{S}\nabla\mathbf{A}\, d\varsigma - \iint u\mathrm{S}\, d\Sigma\mathbf{A},$$

we get [equations (19), (20), § 26]

$$4\pi(\mathbf{A} - \nabla w) = \nabla \iiint u\mathbf{B}\, d\varsigma = -\iiint \mathrm{V}\nabla u\mathbf{B}\, d\varsigma = 4\pi \iiint \mathrm{V}\mathbf{I}\nabla u\, d\varsigma + \iiint \mathrm{V}\mathbf{H}\nabla u\, d\varsigma$$

Also,

$$\iiint \mathrm{V}\mathbf{H}\nabla u\, d\varsigma = \iiint u\mathrm{V}\nabla\mathbf{H}\, d\varsigma - \iint u\mathrm{V}\, d\Sigma\mathbf{H} = 4\pi \iiint u\mathbf{C}\, d\varsigma.$$

This proves equation (34).

That the present theory (and MAXWELL'S), so far as $\mathbf{H}$ and $\mathbf{A}$ depend upon $\mathbf{I}$ and $\mathbf{C}$, thus leads exactly to the ordinary theory is of some importance. One consequence is, that the mechanical action between bodies carrying currents and the induction of currents by the variation of position and magnitude of other currents and magnets, must necessarily be independent of the nature of the medium separating them, so long as that medium is non-magnetic. This is in direct contrast with the known large influence the medium separating two charges of electricity has on the mutual actions of the bodies bearing the charges. On the present, as on MAXWELL'S theory, this is simply owing to the fact that the ordinary theory of magnetism is in the points just mentioned, accurately true, whereas the ordinary theory (action-at-a-distance, with consequently no difference of specific inductive capacity for different media) of electrostatics is not even approximately true. [Whether or not the theory I have called the "ordinary" theory has actually ever been formulated is of little consequence. I have, in the above, accurately enough described what I mean by the term.]

B. *Modifications necessary on account of Hysteresis.*

71. This seems to be the place to consider what bearing the phenomena of hysteresis have upon such theories as the present. No theory of electromagnetism can be considered complete unless it takes this important group of facts into account. I do not here propose to give a theory of hysteresis—so that the present theory must be in this sense confessed incomplete—but it is necessary to notice what modifications ought strictly to be made in the assumptions hitherto adopted.

Professor EWING ('Phil. Mag.,'* V., vol. 30 [1890], p. 205), has given a theory which adapts itself to dynamical methods such as the present. In his theory the phenomena of hysteresis depend upon the fact that groups of molecules can have various stable configurations, different groups at any instant having very different degrees of stability. The stability of a group is liable by variation of $\mathbf{H}$ and Ψ to break down, so that the group takes up another configuration of greater or less stability, and the oscillations which necessarily ensue on the change result to our senses in the production of heat. On this view hysteresis is a phenomenon that prevents us, if we would take full account of the facts, from ignoring certain coordinates we have hitherto ignored. We can, however, go on ignoring these coordinates if we suppose l not to have a constant form in terms of the variables not ignored above, but a form which depends on the particular state as to these groups of molecules of an element of volume. We must, then, suppose certain variables—call them hysteresis-coordinates—which define the relative numbers of groups of different kinds. Of these l will be a function, but they are not of the nature of ordinary dynamical coordinates. Their value merely determines the instantaneous form of l as a function of ordinary coordinates, so that if one or more of the hysteresis-coordinates change, the form of l changes and a new dynamical era begins. In fact, they are very analogous to θ, and like θ they must not be varied when the dynamical coordinates are varied in order to obtain the equations of motion. A mathematical development of Professor EWING's theory may be supposed to furnish the *nature* of these variables, and experiment must then be appealed to at once to test the theory, and if the test be favourable, to find the exact form of l in terms of the variables. And from the mathematical development, or that combined with experiment, we must look to find the laws of variation of the hysteresis coordinates when $\mathbf{H}$ and Ψ vary.

72. This, of course, is only to be looked upon as an ideal procedure of events, which, perhaps, for many years cannot come about. Meanwhile, tentative hypotheses as to the nature of these variables might be made. For instance, it might be assumed that l is always correctly given by equation (1), § 59 above, and that the vector $\mathbf{I}_0$ is the sole hysteresis coordinate. In this case μ (and K ?) would, of course, be assumed a function of $\mathbf{I}_0$ as well as of Ψ. Though this is probably much too

* Or 'Nature,' Oct., 1891, p. 566.

simple a theory for the explanation of all hysteresis phenomena, yet I believe it could be made to account for nearly all the known facts.* But, at present, even if this simple assumption were made, we are very much in the dark as to how $\mathbf{I}_0$ varies with $\mathbf{H}$ and Ψ, and are compelled to fall back on such pure conjectures as are illustrated in the foot-note. To mention only one thing—nearly all the detailed experiments on hysteresis deal only with variations of $\mathbf{H}$ *parallel to itself.*

73. Thus it is useless to attempt a satisfactory theory of hysteresis at present, though we can see vaguely how, perhaps, in the future it may be made to fit into the present theory.

But these considerations show that we must be very cautious in discussing results which depend upon the form of l in terms of $\mathbf{H}$, for they imply that we are very ignorant of this form, even when we know how $\mathbf{I}$ varies with $\mathbf{H}$ under assigned circumstances. Thus, for instance, in equation (30), § 55, we can learn little of the true meaning of ϕ'_0 so far as it depends upon $\mathbf{H}'$. The rest of ϕ' in this equation, however, being independent of the form of l, gives us information of no doubtful character.

C. *On the Strains accompanying these Stresses.*

74. The object of the present paper is to discuss the *general* theory of electromagnetism. It is not proposed, therefore, to deal more than is absolutely necessary in particular problems. A word, however, must be said as to a certain class of problems connected with the stresses just investigated.

After the question of hysteresis has been settled in some such way as just indicated, it will be possible to discuss in detail the exact form of l'_0 of equation (29), § 55 above. To do this, the data on which to argue will generally be the strains which accompany electromagnetic phenomena. This necessitates the consideration of such strains.

75. It must not be supposed that these strains will bear the same relation to the stresses as strains bear to the ordinary stresses considered in the mathematical theory of elasticity. From equations (26), (27), § 50 above, we see that in the case of equilibrium (no external stress)

$$\mathrm{D}_m{}'\nabla'\mathrm{W} - \mathbf{F} = \phi'\Delta', \quad \mathbf{F}_s = [\phi'\mathrm{U}\nu']_{a+b}.$$

* By suitably choosing the form of μ in terms of $\mathbf{I}_0$, and the four functions now to be introduced. Let $|\omega|$, $[\omega]$, $\{\omega\}$ be three positive scalar functions of $\mathrm{T}\omega$, and let $\mathrm{Q}\omega$ be a vector function of the form $\mathrm{T}\omega$ *function* $(\mathrm{U}\omega)$—not in general linear—such that $\mathrm{S}\omega\mathrm{Q}\omega$ is always negative. The form of Q, like that of μ, is a function of $\mathbf{I}_0$. Let $\mathbf{H}$ be the present and $\mathbf{h}$ any previous value of $\mathbf{H}$. Then assume that

$$\dot{\mathbf{I}}_0 = \mathrm{Q}\mathbf{N} \text{ where } \mathbf{N} = |\mathbf{H}|\dot{\mathbf{H}}\int^{\mathbf{H}} [\mathbf{h}]\,\epsilon^{-\int_{\mathbf{h}}^{\mathbf{H}}\{\mathbf{h}\}\,\mathrm{T}d\mathbf{h}}\,\mathrm{T}\,(d\mathbf{h} + \mathrm{U}\mathbf{H}\mathrm{T}d\mathbf{h}),$$

the lower limit of the first integral sign being, strictly, the value of $\mathbf{H}$ at an indefinitely remote epoch, but practically at a time determined by the exponential. I give this merely to show in what *sort* of way we may suppose $\mathbf{I}_0$ to depend on the history of the body.

Taking for simplicity the case where there is no external force (F), or force potential (W), we have

$$\phi'\Delta' = 0, \quad [\phi'U\nu']_{a+b} = 0.$$

Substituting now from equation (27), § 55 for ϕ', we see that this means that there is equilibrium owing to the simultaneous existence of three stresses: (1) the ordinary elasticity theory stress, owing to terms which only involve Ψ; (2) the stress which is independent of l_0 and, therefore, depends *only* on electromagnetic quantities; (3) a stress due to terms in l_0, which involve both Ψ and electromagnetic quantities. When these last are linear in Ψ, the resulting stress will depend, like the second, upon electromagnetic quantities only. If not linear, they will depend both upon Ψ and the electromagnetic quantities. It is quite possible that there should be no strain at all, and yet a very sensible stress due to electromagnetic actions.

In fact, in solving the elasticity problem—having given the distribution throughout the field, of dielectric displacements, of currents, and of magnetisation, required the strain at any point—the only way in which the electromagnetic data can be used is, by finding the force per unit volume and surface respectively due to them, and then treating these forces as external. That is, the knowledge of the *stress* which produces the mechanical effects of electromagnetism is of no use in discovering the strain actually resulting; all the knowledge we can thus utilise is that of the *forces* (per unit volume and surface) due to such stresses. This shows that, to find the true expression for l it is not sufficient to investigate experimentally what strain accompanies a given displacement, or current, or magnetisation at a point.* The problem is much more complicated. The shapes of all the bodies present must be assumed of quite as great importance as the electromagnetic quantities in deciding the form of l from such experiments.

76. These remarks may be illustrated by considering the effect of MAXWELL's stress in two different cases. Choosing one shape of soft-iron body it will be found that the magnetisation will, according to MAXWELL's stress, compress the body; choosing another shape, expansion results.

Suppose we have (1) an anchor ring of soft iron, (2) surrounding this a layer of air of uniform thickness, (3) surrounding this n coils of insulated uniformly distributed wire carrying a current c. Take columnar coordinates r, ϑ, z, the axis of z being the axis of the anchor ring, and let i, j, k be unit vectors (functions of the position of a point) in the directions of dr, $d\vartheta$, dz respectively. At any point inside the coil we have $\mathbf{H}' = 2ncj/r$. Assuming μ' to be a constant scalar $\mathbf{I}' = (\mu' - 1)\,\mathbf{H}'/4\pi$. Hence, from equation (27), § 67,

$$\{\phi'\}\,\Delta' = -\,\Delta'\mathrm{S}\mathbf{I}'\mathbf{H}'/2 = -\,(\mu' - 1)\,\nabla_1{}'\mathrm{S}\mathbf{H}'\mathbf{H}_1{}'/4\pi = -\,n^2c^2\,(\mu' - 1)\,i/\pi r^3.$$

* This seems to be the meaning of the third sentence of the small print on p. 269 of vol. 15., 'Encyc. Brit.,' 9th ed.

At the surface of the soft iron $\mathbf{H}'$ is tangential, and therefore continuous. From this it easily follows that

$$[\{\phi'\}U\nu']_{a+b} = 0.$$

Hence, due to MAXWELL's stress, there is in this case no superficial force and no bodily force in the air, but there is a bodily force in the iron directed towards the axis. The iron will therefore be compressed.

77. For the other case, notice that the force per unit surface due to the electromagnetic part of MAXWELL's stress is $-[\{\phi'\}U\nu']_{a+b}$ and by equation (27), § 67,

$$-8\pi[\{\phi'\}U\nu']_{a+b} = [\mathbf{B}'S\mathbf{H}'U\nu' + \mathbf{H}'S\mathbf{B}'U\nu' - U\nu'\mathbf{H}'^2]_{a+b}.$$

This can be put in several different forms, of which, perhaps, the following are the most useful

$$\left.\begin{aligned} -8\pi[\{\phi'\}U\nu']_{a+b} &= 4\pi[\mathbf{I}'SU\nu'\mathbf{H}']_{a+b} + (\mathbf{H}_t' + \mathbf{B}_n')[SU\nu'\mathbf{H}']_{a+b} \\ &= (4\pi\bar{\mathbf{I}}' + \mathbf{H}_t' + \mathbf{B}_n')[SU\nu'\mathbf{H}']_{a+b} + 4\pi[\mathbf{I}'SU\nu'\bar{\mathbf{H}}']_{a+b} \end{aligned}\right\} \quad . \quad (36),$$

where the bar indicates the mean value for the two regions bounded by the surface, and the suffixes n and t denote normal and tangential components respectively. Thus $\mathbf{B}_n'$ and $\mathbf{H}_t'$ have the same value on both sides of the surface. When $\mathbf{B}'$ is parallel to $\mathbf{H}'$, $\mathbf{B}' = \mu'\mathbf{H}'$ where μ' is a scalar, not necessarily constant. (But if not constant it has here a different meaning from what it has in the rest of this paper.) In this case the tangential component of $-8\pi[\{\phi'\}U\nu']_{a+b}$ is zero. For the first expression of equation (36) gives for the component in question

$$[(4\pi\mathbf{I}_t' + \mathbf{H}_t')SU\nu'\mathbf{H}']_{a+b} = \mathbf{H}_t'[\mu'SU\nu'\mathbf{H}']_{a+b} = \mathbf{H}_t'[SU\nu'\mathbf{B}']_{a+b} = 0.$$

So long then as we deal with magnetically isotropic media this surface traction is normal.

Consider a magnetically isotropic body surrounded by a non-magnetic medium, and let the magnetic region be denoted by the suffix a, so that $\mathbf{I}_b' = 0$. In accordance with what has been just proved we consider only the normal part of the traction. Thus,

$$-8\pi[\{\phi'\}U\nu']_{a+b} = 4\pi[\mathbf{I}'_nSU\nu'\mathbf{H}']_{a+b} + \mathbf{B}'_n[SU\nu'\mathbf{H}']_{a+b}.$$

Let now $[\mathbf{H}'_n]_a = HU\nu'_a$. Thus

$$4\pi[\mathbf{I}'_n]_a = (\mu' - 1)HU\nu'_a, \qquad \mathbf{B}'_n = \mu'HU\nu'_a,$$
$$[SU\nu'\mathbf{H}']_a = -H, \qquad [SU\nu'\mathbf{H}']_{a+b} = (\mu' - 1)H,$$

the last coming from the fact that $[\mathbf{H}'_n]_b = \mathbf{B}'_n$. Thus

$$-8\pi[\{\phi'\}U\nu']_{a+b} = (\mu'-1)^2H^2U\nu'_a;$$

or

$$-[\{\phi'\}U\nu']_{a+b} = -2\pi[I'^2_nU\nu']_a \quad . \quad . \quad . \quad . \quad . \quad . \quad (37).$$

Hence for both paramagnetic and diamagnetic isotropic bodies surrounded by non-magnetic media, MAXWELL's stress leads to a surface traction which is always a tension (except as in the anchor-ring when it is zero).

78. Consider now the well-known ordinary case of a soft-iron ellipsoid (μ' a constant scalar) brought into a uniform field. Inside the ellipsoid $\mathbf{B}'$, $\mathbf{H}'$, and $\mathbf{I}'$ are all constant, and therefore $\{\phi'\}\Delta' = 0$, so that there is no bodily force. Also since [equation (27), § 67]

$$\{\phi'\}\Delta' = V\mathbf{C}'\mathbf{B}' - \nabla'_1 S\mathbf{I}'\mathbf{H}'_1 + V\nabla' V\mathbf{I}'\mathbf{H}'/_2 \quad . \quad . \quad . \quad . \quad . \quad (38),$$

there is no bodily force in the surrounding medium. Hence, in the present case, the only force is the tension at the surface. The ellipsoid will therefore be expanded by MAXWELL's stress.

D. *Thermoelectric, Thermomagnetic, and* HALL *Effects.*

79. It will be found convenient to discuss these various effects together.

The natures of the thermoelectric and HALL effects are well known and need no description here. The thermomagnetic effects are perhaps not so well-known. The original papers of VON ETTINGSHAUSEN and NERNST (the discoverers of these effects) are in 'Wied. Ann.,' xxxi. (1887), 737 and 760, xxxiii. (1888), 126, 129, 474. The effects are briefly described in Professor J. J. THOMSON's 'Applications of Dynamics to Physics and Chemistry,' 1st ed., § 57. The principal features of these effects are that the electromotive forces due to differences of temperature are modified in two ways by the presence of magnetic force. First, parallel to Θ there is an electromotive force that varies approximately as $\mathbf{H}^2T\Theta$ (the "longitudinal" thermomagnetic effect); and, secondly, at right angles to both Θ and $\mathbf{H}$ there is an electromotive force $BV\Theta\mathbf{H}$, where B is a scalar dependent on the temperature, but approximately independent of $T\Theta$ and $T\mathbf{H}$ (the "transversal" thermomagnetic effect). The latter effect is especially large in bismuth. There is evidence that these effects are closely connected with the HALL effect.

80. The natural way to discuss these results would appear to be to attempt to explain them by suitable terms in l. But on the present theory it is possible that they may be explained by terms in x. According to the first explanation they would be reversible phenomena, and according to the second irreversible phenomena involving dissipation of energy. Thermoelectric effects are certainly at present looked upon by physicists as reversible phenomena.

The two explanations—which will for the future be referred to as the theory of reversibility and the theory of irreversibility respectively—will be found in many respects very analogous, though, of course, we must expect some striking difference of results. On the theory of this paper the most striking would seem to be that, while thermoelectric effects must in the main be explained on the theory of reversibility, the explanation of the thermomagnetic effects by this theory is inadmissible by reason of certain collateral consequences.

81. Let ϖ—which has no connection with the ϖ of § 54 above—be a linear vector function of a vector, itself a function of θ, Ψ, and $\mathbf{H}$; and let

$$\varpi = [0] + [1] + [2] = \Sigma [n] \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (1),$$

where $[n]$ is a homogeneous function of degree n in the vector $\mathbf{H}$.

In particular, let ϖ be given by

$$\varpi\omega = \mathrm{A}\omega + \mathrm{BV}\omega\mathbf{H} - \omega\mathrm{S}\mathbf{H}\mathrm{C}\mathbf{H} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (2),$$

where A, B, C are linear vector functions of a vector, themselves functions of Ψ and θ only. B and C, but not A, may for simplicity be assumed self-conjugate. Notice that $\mathrm{S}\mathbf{H}_{\mathrm{H}}\nabla . [n] = - n [n]$, and therefore

$$(1 + \mathrm{S}\mathbf{H}_{\mathrm{H}}\nabla.)\, \varpi = \Sigma (1 - n) [n] = \mathrm{A} + \mathrm{S}\mathbf{H}\mathrm{C}\mathbf{H} \quad . \quad . \quad . \quad . \quad . \quad (3).$$

Similarly,

$$(\mathrm{S}\mathbf{K}_{\mathrm{K}}\nabla. + \mathrm{S}\mathbf{H}_{\mathrm{H}}\nabla. + \mathrm{S}\Theta_{\Theta}\nabla.)\, \mathrm{S}\mathbf{K}\varpi\Theta = - \mathrm{S}\mathbf{K} (2 [0] + 3 [1] + 4 [2])\, \Theta \quad . \quad (4),$$

ϖ and A will be assumed to be of a class given by

$$\left.\begin{aligned} \mathrm{S}\tau\varpi\sigma d\varsigma = \mathrm{S}\tau'\varpi'\sigma' d\varsigma' = \mathrm{S}\tau''\varpi''\sigma'' d\varsigma' \\ \varpi' = \chi'^{-1}\varpi\chi', \qquad \varpi'' = \psi^{-1}\varpi\psi \end{aligned}\right\} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (5),$$

and, of course, exactly similarly for A. It should be noticed that unlike the two classes of § 9 above, ϖ and A have not the property that if ϖ or A is self-conjugate, so also is ϖ' or A$'$ and ϖ'' or A$''$. But they have another simple property, namely, that if ϖ is a scalar,

$$\varpi = \varpi' = \varpi''. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (6).$$

It may also be noticed that $\varpi\sigma$ is an intensity, and $\varpi_c\tau$ a flux where ϖ_c stands for the conjugate of ϖ.

B is assumed to be of Class II. of § 9 above, and C of a class given by

$$\left.\begin{aligned} \mathrm{S}\sigma_a\mathrm{C}\sigma_b = \mathrm{S}\sigma_a'\mathrm{C}'\sigma_b' = \mathrm{S}\sigma_a''\mathrm{C}''\sigma_b'' \\ \mathrm{C}' = \chi\mathrm{C}\chi', \quad \mathrm{C}'' = \psi\mathrm{C}\psi \end{aligned}\right\} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (7),$$

so that it is very closely allied to Class I. of § 9 above.

It will now be seen that to obtain ϖ' or ϖ'' from ϖ it is only necessary to change A, B, C, and **H** into A$'$, B$'$, C$'$, and **H**$'$, or into A$''$, B$''$, C$''$, and **H**$''$ respectively. The statement is obvious so far as A, C, and **H** are concerned. With regard to B, we have, by §§ 7, 9,

$$\begin{aligned} B'V\omega\mathbf{H}' &= m\chi'^{-1}B\chi^{-1}V\omega\chi'^{-1}\mathbf{H} \\ &= \chi'^{-1}BV\chi'\omega\mathbf{H} \text{ [Tait's `Quaternions,' 3rd ed., § 157, eq. (2)]}, \end{aligned}$$

which, with equation (5), proves the statement.

82. For the theory of reversibility, it is assumed that l contains a term g given by $g = -\,S\mathbf{D}\varpi\Theta$. For the theory of irreversibility, it is assumed that x contains a term g given by $g = -\,S\mathbf{K}\varpi\Theta$. Denote the various parts of **E**, f &c., depending upon g by the suffix g. It conduces to clearness to arrange the general results of these two assumptions in parallel columns thus:—

Theory of Reversibility.

l contains a term g given by

$$g = -\,S\mathbf{D}\varpi\Theta \quad . \quad . \quad . \quad . \quad (8).$$

This contributes terms $\mathbf{E}_g$, $\mathbf{E}_{sg}$ to the right of equations (29), (31), § 50, given by

$$\mathbf{E}_g = \varpi\Theta\,, \quad \mathbf{E}_{sg} = 0 \quad . \quad . \quad . \quad (9).$$

In this is not included the part of $\dot{\mathbf{A}}$ due to g, but this is practically given by equations (15), (16), below. By equation (11), § 46,

$$\lambda_g = S\mathbf{D}\,(1 + S\mathbf{H}_H\nabla.)\,\varpi\Theta \quad . \quad . \quad (10).$$

By equations (11), (12), § 34 (putting ϖ_θ for $\partial\varpi/\partial\theta$)

$$\begin{aligned} f_g = -\,&S\mathbf{D}\,(1 + S\mathbf{H}_H\nabla.)\,\varpi_\theta\Theta \\ &+ S\mathbf{D}\,(1 + S\mathbf{H}_H\nabla.)\,\varpi\Delta \quad . \quad (11). \end{aligned}$$

$$f_{sg} = -\,[S\mathbf{D}\,(1 + S\mathbf{H}_H\nabla.)\,\varpi U\nu]_{a+b} \quad . \quad . \quad . \quad (12).$$

Hence to the *left* of equations (35), (36), § 40, are contributed for a steady field

$$\begin{aligned} \theta\dot{f}_g = \theta\{-\,&S\mathbf{C}\,(1 + S\mathbf{H}_H\nabla.)\,\varpi_\theta\Theta \\ &+ S\mathbf{C}_1\,(1 + S\mathbf{H}_{1H}\nabla.)\,\varpi_1\nabla_1\} \quad . \quad (13). \end{aligned}$$

$$\theta\dot{f}_{sg} = -\,\theta\,[S\mathbf{C}\,(1 + S\mathbf{H}_H\nabla.)\,\varpi U\nu]_{a+b} \quad . \quad . \quad (14).$$

By equations (34), § 50

$$\begin{aligned} \mathbf{B}_g &= -\,4\pi_H\nabla S\mathbf{D}\varpi\Theta \\ &= 4\pi\,\{VB\mathbf{D}\Theta - 2C\mathbf{H}S\mathbf{D}\Theta\} \quad . \quad . \quad (15). \end{aligned}$$

Theory of Irreversibility.

x contains a term g given by

$$g = -\,S\mathbf{K}\varpi\Theta \quad . \quad . \quad . \quad . \quad (8\text{A}).$$

This contributes terms $\mathbf{E}_g$, $\mathbf{E}_{sg}$ to the right of equations (29), (31), § 50, given by

$$\mathbf{E}_g = -\,\varpi\Theta\,, \quad \mathbf{E}_{sg} = 0 \quad . \quad . \quad (9\text{A}).$$

In this is not included the part of **a**, due to g, but this is practically given by equation (16A) below.

Contributed to the *right* of equations (35), (36), § 40, are terms (for any field, steady or not) given by

$$\begin{aligned} \theta\dot{f}_g = -\,&S\mathbf{K}\,(2\,[0] + 3\,[1] \\ &+ 4\,[2])\,\Theta - \theta S\mathbf{K}_1\varpi_1\nabla_1 \quad . \quad . \quad (13\text{A}). \end{aligned}$$

$$\theta\dot{f}_{sg} = \theta\,[S\mathbf{K}\varpi U\nu]_{a+b} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (14\text{A}).$$

Theory of Reversibility.

Hence for a steady field

$$\dot{\mathbf{B}}_g = 4\pi\,(\mathrm{V}\mathbf{B}\mathbf{C}\Theta - 2\mathbf{C}\mathbf{H}\mathrm{S}\mathbf{C}\Theta) \quad . \quad (16).$$

and if this is the only part of $\dot{\mathbf{B}}$ for a steady field, we have

$$0 = \mathrm{S}\nabla\dot{\mathbf{B}}_g = 4\pi\mathrm{S}\nabla\,(\mathrm{V}\mathbf{B}\mathbf{C}\Theta - 2\mathbf{C}\mathbf{H}\mathrm{S}\mathbf{C}\Theta) \quad . \quad . \quad (17).$$

$$0 = [\mathrm{SU}\nu\dot{\mathbf{B}}_g]_{a+b} = 4\pi\,[\mathrm{SU}\nu\,(\mathrm{V}\mathbf{B}\mathbf{C}\Theta - 2\mathbf{C}\mathbf{H}\mathrm{S}\mathbf{C}\Theta)]_{a+b} \quad . \quad . \quad . \quad (18).$$

The following equations will be explained below :—

$$\sigma = \theta\,(\partial \mathrm{P}/\partial\theta - d\mathrm{P}/d\theta) \quad . \quad . \quad (19).$$

$$\Pi = \theta\,[\mathrm{P}]_{a-b} \quad . \quad . \quad . \quad . \quad . \quad . \quad (20).$$

$$\frac{d(\Pi/\theta)}{d\theta} + \frac{[\sigma - \theta \mathrm{P}_\theta]_{a-b}}{\theta} = 0 \quad . \quad . \quad (21).$$

$$\left.\begin{aligned} \mathrm{E} &= -\int \mathrm{S}\,d\rho\mathrm{E}_g = \int_{\theta_0}^{\theta} (\Pi/\theta)\,d\theta \\ &= \Pi - \Pi_0 + \int_{\theta_0}^{\theta} [\sigma - \theta\mathrm{P}_\theta]_{a-b}\,d\theta \end{aligned}\right\} . \quad (22).$$

Contributed to equation (27) § 55 we have

$$\phi_g' = 2m^{-1}\mathrm{S}\mathbf{D}\varpi_1\Theta.\chi\mathbb{C}_1\chi' \quad . \quad . \quad (23).$$

Hence, for a steady field,

$$\phi_g' = 2m^{-1}\mathrm{S}\mathbf{C}\varpi_1\Theta.\chi\mathbb{C}_1\chi' \quad . \quad . \quad (24).$$

Theory of Irreversibility.

For any field, steady or otherwise,

$$\mathfrak{b}_g = 4\pi\,(\mathrm{V}\mathbf{B}\mathbf{K}\Theta - 2\mathbf{C}\mathbf{H}\mathrm{S}\mathbf{K}\Theta) \quad . \quad (16\mathrm{A})$$

and if g is the only term in x containing $\mathbf{H}$, we have

$$0 = \mathrm{S}\nabla\mathfrak{b}_g = 4\pi\mathrm{S}\nabla\,(\mathrm{V}\mathbf{B}\mathbf{K}\Theta - 2\mathbf{C}\mathbf{H}\mathrm{S}\mathbf{K}\Theta) \quad . \quad (17\mathrm{A}).$$

$$0 = [\mathrm{SU}\nu\mathfrak{b}_g]_{a+b} = 4\pi\,[\mathrm{SU}\nu\,(\mathrm{V}\mathbf{B}\mathbf{K}\Theta - 2\mathbf{C}\mathbf{H}\mathrm{S}\mathbf{K}\Theta)]_{a+b} \quad . \quad . \quad . \quad (18\mathrm{A}).$$

The following equations will be explained below :—

$$\sigma = -\,(2\mathrm{A} + \theta\,d\mathrm{A}/d\theta) \quad . \quad . \quad (19\mathrm{A}).$$

$$\Pi = \theta\,[\mathrm{A}]_{a-b} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (20\mathrm{A}).$$

$$\frac{d(\Pi\theta)}{d\theta} + \theta\,[\sigma]_{a-b} = 0 \quad . \quad . \quad . \quad (21\mathrm{A}).$$

$$\left.\begin{aligned} \mathrm{E} &= -\int \mathrm{S}\,d\rho\mathrm{E}_g = -\int_{\theta_0}^{\theta} (\Pi/\theta)\,d\theta \\ &= \Pi - \Pi_0 + \int_{\theta_0}^{\theta} [\sigma]_{a-b}\,d\theta \end{aligned}\right\} \quad (22\mathrm{A}).$$

83. Before discussing these equations in detail, it will be shown how in the theory of reversibility certain very important restrictions must be imposed on the generality of ϖ as a function of θ, Ψ, and $\mathbf{H}$, in order that known experimental facts shall not be contradicted. These restrictions seem, for the most part, to depend on the particular form of theory adopted in this paper; but as the particular features of the theory which are thus involved are held by many physicists, it is of interest to notice exactly what part of our fundamental assumptions causes the restrictions. The direct cause of the trouble is the equation $\mathbf{B} = 4\pi_{\mathrm{H}}\nabla l$. Now, it will be remembered (§ 16 above) that this equation flows from the assumptions (1) that l contains $\mathbf{H}$ and not $\mathbf{B}$ explicitly, and (2) that $4\pi\mathbf{C} = \mathrm{V}\nabla\mathbf{H}$. If, then, it can be proved experimentally that the consequences of the restrictions developed below (the chief of which is that thermomagnetic phenomena involve dissipation of energy, and that a part of the thermoelectric phenomena do the same), are contrary to fact; one or both of these assumptions must be relinquished. Thus, perhaps, in this unexpected quarter, will be found a practical test of the truth of MAXWELL's fundamental assumption $4\pi\mathbf{C} = \mathrm{V}\nabla\mathbf{H}$. The consequences of relinquishing either of the above assumptions would probably be much the same, since it would lead to the physicist being compelled to recognise with

Professor J. J. THOMSON ['Applications,' § 17 (4)] magnetic coordinates independently of electric coordinates. It is interesting to note in this connection that Professor J. J. THOMSON (*ibid.*, § 59) working on somewhat different lines from this paper, has also found that thermomagnetic phenomena have a distinct bearing on the equation $4\pi\mathbf{C} = V\nabla\mathbf{H}$. His conclusion is that this equation must be given up, and that instead we shall have

$$V\nabla\mathbf{H} = 4\pi\mathbf{C} + (4\pi/3)\,B\,(\Theta S\nabla d - S\Theta\nabla\,.\,d),$$

B being here assumed to be a scalar. He assumes that thermomagnetic phenomena are reversible.

84. The first equation that challenges attention is (16). It might be thought that it was a truism that $\mathbf{B}$ should remain constant in a steady field. This, however, is not the case. If the steady increase of $\mathbf{B}$ implied by this equation does not produce a steady increase in some physical quantity which can be measured directly, the field will remain steady in the ordinary sense though $\mathbf{B}$ increases. Now the physically measurable phenomena depending on $\mathbf{B}$ can be conveniently divided into three groups, (1) the stress at the point — which leads to mechanical phenomena capable of measurement, (2) the effect it has in modifying the value of $\mathbf{H}$ at all points of the field — which again leads to mechanical phenomena capable of measurement, (3) its effect on the induction of currents. As to (1) it is certainly true that in many instances above $\mathbf{B}$ does occur in the expression for the stress at a point, but this is in the case of solids purely a mathematical result. By equation (27) § 55 we see that the rate of variation of $\mathbf{B}$ will in general have no effect on the stress at a point. With regard to (2) the question must be asked, how does the value of $\mathbf{B}$ affect the values of $\mathbf{H}$, $\mathbf{C}$, &c., at points other than that considered? The answer is — solely by reason of the equations $S\nabla\mathbf{B} = 0$, $[SU\nu\mathbf{B}]_{a+b} = 0$, where it must be remembered $\mathbf{B}$ is an explicit function of $\mathbf{H}$, $\mathbf{C}$, &c. If then the rates of variation $S\nabla\mathbf{B}$ $[SU\nu\mathbf{B}]_{a+b}$ due to g are zero, the steady increase of $\mathbf{B}_g$ will not produce a steady increase of $\mathbf{H}$, $\mathbf{C}$, &c., at any point of space. But these conditions are insured by equations (17), (18). Hence we see that $\dot{\mathbf{B}}_g$ need not cause time variation of any mechanical phenomena. As to (3) the only electric effect of $\mathbf{B}$ (due to $-\dot{\mathbf{A}}$ in $\mathbf{E}$) is one which remains constant so long as $\dot{\mathbf{B}}$ remains constant, and, therefore, does not affect the steadiness of the field. Hence equation (16) presents no difficulty.

Equations (15), (17), and (18), however, do present very formidable difficulties. It has been stated that the influence of $\mathbf{B}$ on the electromagnetic quantities of the field is due to the equations $S\nabla\mathbf{B} = 0$, $[SU\nu\mathbf{B}]_{a+b} = 0$. It must now be added that the manner in which it thus affects the field depends on the form of its expression in terms of $\mathbf{H}$, $\mathbf{C}$, &c. Now unless the principal term in $\mathbf{B}$ is one depending on $\mathbf{H}$ only, the behaviour of the body in question would not at all approximate to the magnetic behaviour we know that bodies exhibit. For instance, we know that bismuth always behaves very approximately as if it were non-magnetic. This requires that the principal term in $\mathbf{B}'$ should be $\mu_0\mathbf{H}'$. It is needless to say that this will not in general be the case if $\mathbf{B}$ involves $\mathbf{D}$. In fact, in a steady field, the only conditions we should be able to assert that $\mathbf{B}$ imposed upon the field would be those which resulted from equations (17), (18), which do not contain $\mu_0\mathbf{H}'$ at all.

85. The conclusion we are bound to come to is that on the present theory there must be no term in l which will result in ${}_{\mathrm{H}}\nabla l$ containing $\mathbf{D}$. Thus the thermomagnetic effects must be left to the theory of irreversibility.

It might be thought that all difficulty would vanish if, in g, instead of $\mathbf{D}$ we substituted $\mathbf{d}$, since this would involve terms in $\mathbf{e}$ similar to those given above for $\mathbf{E}$ and since $\mathbf{E} = -\mathbf{e}$. This, however, is not the case. No term in $\mathbf{e}$ can affect steady currents. The equation $\mathbf{E} + \mathbf{e} = 0$ determines the value of $\mathbf{d}$ but does not affect the value of $\mathbf{C}$. Again, since in equations (13) (14) we should have $\mathbf{c}$ instead of $\mathbf{C}$, we see that no thermal effects would, owing to g, occur in a steady field.

86. None of these difficulties and restrictions are met with in the theory of irreversibility. We shall find, however, that the thermoelectric consequences of that theory are inconsistent with known facts. Both theories, therefore, are assumed as true in part.

In connection with thermoelectricity it is necessary first to establish equations (19) to (22) and (19A) to (22A). In these equations the notation adopted is that of Professor CHRYSTAL's article in the 'Encyc. Brit.,' 9th ed., vol. viii., p. 97. He there considers a circuit of two unstressed isotropic metals a and b, one of the junctions being at temperature θ, and the other at temperature θ_0. The positive direction round the circuit is taken as that from the metal a to the metal b at the junction θ. Π, σ* are the PELTIER and THOMSON effects at the temperature θ, and Π_0 the PELTIER effect at the temperature θ_0. E is the electromotive force round the circuit due to thermoelectric effects.

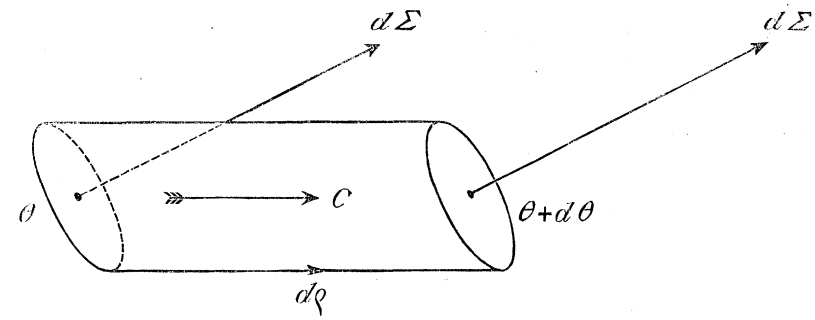
To consider such a case as this, B and C are, of course, ignored, and A is supposed a scalar. For the future, in order to distinguish more clearly between the theories, P will be substituted for A on the theory of reversibility. It is necessary now to distinguish between $\partial/\partial\theta$ and $d/d\theta$. The former denotes differentiation when θ, Ψ are taken as independent variables. Now in all the commoner experiments on thermoelectricity, different parts of the circuit are not similarly strained, but similarly stressed. This may be taken account of by regarding Ψ as a function of θ. Regarding it as such $d/d\theta$ denotes *total* differentiation with regard to θ.

87. σ, the THOMSON effect, is defined by saying that the heat "absorbed" by the metal between two sections at temperature θ and $\theta + d\theta$, while a unit quantity of electricity passes in the direction from the first to the second section, is $\sigma d\theta$, or the rate at which heat is absorbed in this part equals the rate at which electricity is flowing

* There seems no danger in using σ here for this *scalar*, though in the rest of the present paper it is taken as the type of an intensity (*vector*), nor in using E here for the electromotive force round the circuit, though in the rest of the paper it stands for intrinsic energy.

in the assigned direction $\times \sigma d\theta$. Consider an elementary (generally oblique) cylinder whose generating lines are parallel to **C** or **K** and whose faces are coincident with the isothermal surfaces θ and $\theta + d\theta$. Let $d\rho$ be the vector in the direction from the section θ to the section $\theta + d\theta$, representing a generating line, and $d\Sigma$ the vector area of either face, drawn inwards at the section θ and outwards at the section $\theta + d\theta$ (fig. 2). The rate of "absorption" of heat means the rate at which energy dis-

Fig. 2.



appears as heat (positive when it causes *fall* of temperature) and appears as some other form of energy, in the present case that of "electrical separation." Taken per unit volume of the standard position of matter this $= \theta \times$ what is contributed to the *left* of equation (35), § 40. [The truth of this statement should be clearly recognised. The principal term on the left of equation (35), § 40, is $c\dot{\theta}/\theta$ where c is the capacity for heat per unit volume. Any other positive term contributed to this side will therefore tend to render $\dot{\theta}$ negative.] Hence the rate of absorption of heat per unit volume due to the terms now under consideration will be + the expression on the right of equation (13), § 82, and − the expression on the right of equation (13A). Putting $\varpi =$ the scalar P or A, and noting that $S\nabla \mathbf{C} = 0$, and that for a steady field $S\nabla \mathbf{K} = 0$, we see that on the theory of reversibility the rate of absorption of heat for our element $- Sd\Sigma d\rho$ of volume is

$$\theta\,(-\,P_\theta S\mathbf{C}\nabla\theta + S\mathbf{C}\nabla P)\,(-\,Sd\Sigma d\rho)$$

and on the theory of irreversibility

$$(2AS\mathbf{K}\nabla\theta + \theta S\mathbf{K}\nabla A)\,(-\,Sd\Sigma d\rho).$$

Now $\nabla P = \nabla\theta \,.\, dP/d\theta$ and $\nabla A = \nabla\theta \,.\, dA/d\theta$, and since **C** is parallel to $d\rho$, these two may be interchanged in the expressions just given. Thus the rates of absorption of heat on the theories of reversibility and irreversibility respectively, are

$$-\,\theta\,(-\,\partial P/\partial\theta + dP/d\theta)\,Sd\rho\nabla\theta S\mathbf{C}\,d\Sigma$$

and

$$-(2A+\theta\, dA/d\theta)\, Sd\rho\nabla\theta SC\, d\Sigma.$$

But the rate of absorption of heat also $= \sigma\, d\theta \times$ the rate of flow through the element from θ to $\theta + d\theta$

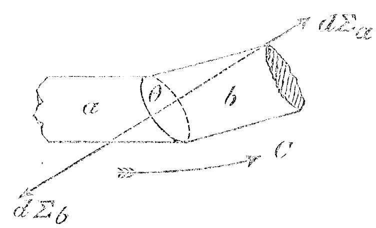
$$= \sigma S\, d\rho\nabla\theta SC\, d\Sigma.$$

Equating these different expressions for the same quantity, we get equations (19) and (19A).

Equations (20) and (20A) are obtained in an exactly similar way. It need only now be said that from equation (14) the rate of absorption of heat at an element of the boundary of the metals a and b [fig. 3] is

$$\theta\,[PSC\, d\Sigma]_{a+b} = -\theta\,[P]_{a-b}SC\, d\Sigma_a,$$

Fig. 3.



and also that by the definition of the PELTIER effect Π, it is $= \Pi \times$ the rate of flow from the metal a to the metal b

$$= -\Pi SC\, d\Sigma_a,$$

from which equation (20) follows. Similarly for equation (20A). Equations (21) and (21A) are easily deduced from those now established.

88. In connection with equations (22) and (22A) it is advisable to make what may be looked upon as a digression, to examine whether, on the present theory, we have a right to identify the line integral of any part of $\mathfrak{E}$ round a circuit with what, in the laboratory, is known as the electromotive force of the particular kind round it. To test this, we must see whether the total line integral of RK round the circuit = what is called the whole resistance of the circuit $\times$ the whole current.

In equations (29), (31), § 50 occurs a scalar $y + Y$ or v which, unlike the other terms in the equations, does not depend directly or indirectly (as is the case with $d\mathbf{A}/dt + \mathbf{a}$) upon the form of l or x. Consider now any closed curve which, if it anywhere crosses a surface of discontinuity, passes, we shall suppose, from the region a to the region b. Then, in the expression $-\int S\mathfrak{E}\, d\rho - \Sigma S\mathfrak{E}_s U\nu_a$ this unknown scalar does not appear as can be easily seen by equations (29), (31). Before taking this line integral, remove RK to the left of equation (29), § 50, keeping all the other terms on the right.

It is the line integral (including the terms contributed by $\Sigma \mathrm{SE}_s \mathrm{U}\nu_a$*) that thus appears on the right, which is ordinarily called the total electromotive force round the curve.*

Let us examine whether this statement is consistent with the one above about whole current and whole resistance. Suppose the motion steady so that **K** obeys the laws of incompressibility. Consider an infinitely small tube of flow, and let this be the line along which the integral is taken. Let c be the whole current flowing along the tube. Consider an elementary right section of the tube of length $\mathrm{T}d\rho$, and cross-section $\mathrm{T}d\Sigma$. Thus

$$\mathrm{K} = c\mathrm{UK}/\mathrm{T}d\Sigma, \quad d\rho = \mathrm{UKT}d\rho,$$

and, therefore, the part contributed to the integral $-\int \mathrm{S}d\rho\mathrm{RK}$ by the element in question is

$$-c\mathrm{SUKRUK}\,.\,\mathrm{T}d\rho/\mathrm{T}d\Sigma.$$

If then we choose to define as follows: (1) $-\mathrm{SUKRUK}$ = the specific resistance of the body at the point, (2) specific resistance $\times\ \mathrm{T}d\rho/\mathrm{T}d\Sigma$ = the resistance of the element, (3) the sum of the resistances of all the elements = the whole resistance of the tube, we shall have

$$-\int \mathrm{S}d\rho\mathrm{RK} = \textit{current flowing along tube} \times \textit{whole resistance of tube},$$

or

$$\textit{conductivity of tube} \times (-\int \mathrm{S}d\rho\mathrm{RK}) = \textit{current flowing along tube},$$

which defines the conductivity as the reciprocal of the whole resistance. Now split any finite tube of flow into an infinite number of such elementary tubes, call the sum of the conductivities of the elementary tubes the whole conductivity of the finite tube, and call the reciprocal of this last the whole resistance of the finite tube. We shall then have that the *mean* of the values of $-\int \mathrm{S}d\rho\mathrm{RK}$ for the elementary tubes = the whole resistance of the finite tube × the whole current along it. All this may, I think, be said to be in complete agreement with the ordinary theory, but it serves to call attention to the fact—important in connection with the longitudinal effect mentioned in § 79 above—that anything which interferes with the ordinary lines of flow will alter the apparent resistance.

89. To return to our immediate purpose, we are now at liberty to say that the line integral of any term contributed to the right of equation (29), § 50, round a closed circuit implies an equal electro-motive force round the circuit in the ordinary sense. Equations (22) and (22A) are easily seen to follow.

Comparing, now, equations (20), (21), (22), (20A), (21A), and (22A), with equations (4), (5), (6), and (7) on p. 97, vol. 8, 'Encyc. Brit.,' 9th ed., we see the results of the theory of reversibility only differ from the ordinary theory in having $\sigma - \theta \mathrm{P}_\theta$ in place of σ, while those of the theory of irreversibility differ widely.

Thus the former theory explains thermoelectric effects satisfactorily. But we shall see that we cannot suppose A zero. Hence we must on the present theory suppose that the main thermoelectric effects are reversible, but that there are subsidiary irreversible ones that with the present means of experiment it would be practically impossible to disentangle from the former.

90. The detailed comparison between the two theories is most clearly made by means of the thermoelectric diagram. From equation (22A) we see that on the theory of irreversibility the thermoelectric power, instead of being Π/θ, is $-\Pi/\theta$. To any one who is acquainted with the ordinary thermoelectric diagram, the following statements will be sufficiently obvious from the accompanying figures :—

Theory of Reversibility.

Abscissa $= \theta$.

Ordinate $= -$ thermoelectric power with respect to lead

$$= -\frac{\Pi\,(\text{lead})}{\theta} = -\mathrm{P}.$$

$\mathrm{E} =$ the area marked in fig. 4.

$\sigma - \theta \mathrm{P}_\theta = \mathrm{PS}$ of fig. 5.

$\Pi = +$ area marked in fig. 6.

$\Pi - \Pi_0 = +$ area marked in fig. 7.

$\int_{\theta_0}^{\theta} [\sigma - \theta \mathrm{P}_\theta]_{a-b}\, d\theta = -$ area marked in fig. 8.

Theory of Irreversibility.

Abscissa $= \theta$.

Ordinate $= -$ thermoelectric power with respect to lead

$$= +\frac{\Pi\,(\text{lead})}{\theta} = +\mathrm{A}.$$

$\mathrm{E} =$ the area marked in fig. 4.

$\sigma = -3\mathrm{PR}$ of fig. $5 = -2\mathrm{PQ} - \mathrm{PS}$.

$\Pi = -$ area marked in fig. 6.

$\Pi - \Pi_0 = -$ area marked in fig. 7.

$\int_{\theta_0}^{\theta} [\sigma]_{a-b}\, d\theta = +$ (area marked in fig. 8).

$+ 2$ (area marked in fig. 4).

$= +$ (area marked in fig. 7).

$+$ (area marked in fig. 4).

Fig. 4.

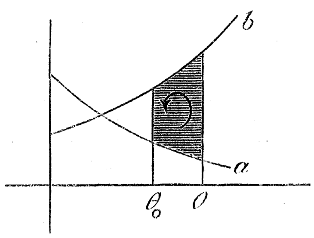
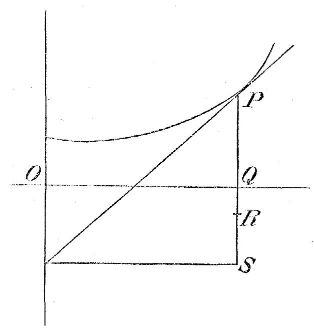


Fig. 5.



[In this figure OQ is the axis and QS = 3QR.]

Fig. 6.

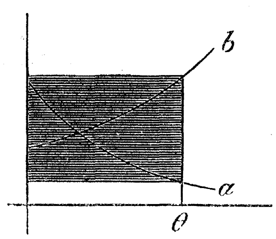


Fig. 7.

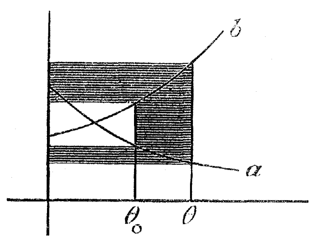
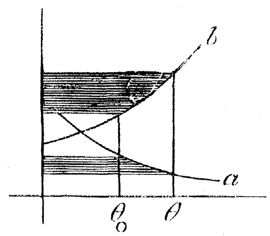


Fig. 8.



The following results may be noted :—

No series of experiments confined to determinations of the electromotive forces resulting from differences of temperature at the junctions of thermoelectric circuits can distinguish between the two theories.

On the theory of reversibility the following statement is true; on the theory of irreversibility the contrary is true. In a thermoelectric circuit of two metals, if a galvanic current be passed across the junction in the same direction as that of the current that would be produced by heating the junction, the effect is absorption, and vice versâ [TAIT'S 'Heat,' 1st ed., § 192]. In many cases this statement has been verified experimentally, and no one, I think, has ever asserted that he has obtained the contrary. Hence the main thermoelectric effects *cannot* be explained on the theory of irreversibility.

On the theory of reversibility $\sigma - \theta P_\theta$ *takes the place of* σ *in the ordinary theory.*

91. This last statement is of importance in connection with a difficulty purposely passed over till now. By equation (23) § 82 it appears that $\mathbf{D}$ enters into the expression for the stress unless P be independent of the strain. Thus the stress depends on the electric history of the substance. This involves difficulties of two kinds. First it shows that in all bodies for which P is not thus independent the stress would probably be widely different from what it is ordinarily assumed to be. This, however, would not affect the apparent mechanical effect on a conductor as a whole since, as already noticed, that effect depends upon the stress just outside the conductor, *i.e.*, the stress in the surrounding *dielectric* in which $\mathbf{D}$ does not increase indefinitely. But one would think that its effects on the mutual behaviour of different parts of a conductor would have been observed. The second difficulty is connected with equation (24). It might be thought a truism that ϕ' should remain constant in a steady field. But as with $\mathbf{B}_g'$ (§ 84) this is not the case. We saw in § 75 that so long as any term which contributes to ϕ' contributes zero to $\phi'\Delta'$ and $[\phi'U\nu']_{a+b}$, it produces no effect on the motion and strains of bodies, that is, no mechanical effect whatever. The conditions for a steady field, so far as stress is concerned, are, therefore, that $\dot{\phi}'\Delta' = 0$ and $[\dot{\phi}'U\nu']_{a+b} = 0$. If then g is the only term in l containing $\mathbf{D}$, equation (24) gives for a steady field

$$m^{-1}S\mathbf{C}\varpi_1\Theta \,.\, \chi\mathrm{C}_1\chi'\Delta' = 0,\ [m^{-1}S\mathbf{C}\varpi_1\Theta \,.\, \chi\mathrm{C}_1\chi'U\nu']_{a+b} = 0 \quad . \quad . \quad . \quad (25).$$

To see what *may* be the approximate effect of this, let us assume ϖ to be a scalar (P) whatever be the strain, and let the scalar be a function of the temperature and the density of the body only. Then

$$\begin{aligned} \dot{\phi}_g' &= 2m^{-1}S\mathbf{C}\Theta \,.\, \chi\mathrm{C}D_m'\chi' \,.\, \partial P/\partial D_m' \\ &= 2D_m m^{-1}S\mathbf{C}\Theta \,.\, \chi\mathrm{C}\,(m^{-1})\,\chi' \,.\, \partial P/\partial D_m' \ [\S\ 49] \\ &= -\ D_m'S\mathbf{C}'\Theta' \,.\, \partial P/\partial D_m' \ [(10)\ \S\ 54 \text{ and II. } \S\ 8]. \end{aligned}$$

From the first of equations (25) it follows that the scalar $\mathbf{SC}'\Theta' D_m' \partial P/\partial D_m'$ must be constant throughout any single conductor; and from the second, that it must be constant throughout any number of conductors in contact, and if the conductors are anywhere bounded by a dielectric, *i.e.*, *invariably*, this constant value must be zero. Hence with present assumptions $\mathbf{SC}'\Theta'$ must be zero everywhere in a steady field. [This is not quite accurate since if P, regarded as a function of D_m' is a maximum or minimum, $\partial P/\partial D_m' = 0$. It does not seem hopeful to pursue this supposition however.] This presents no difficulty in ordinary cases, since $\mathbf{C}$ and Θ—to drop the dashes as no longer necessary—would generally be very approximately at right angles in any case. If, however, we contemplate such a case as the attempt by ordinary means to force a galvanic current and a stream of heat in the same direction through a conductor, some very curious consequences are involved. The most obvious of them seem to be that both the heat and electric apparent conductivities would be *largely* altered. That no such *large* alterations in these physical quantities have been observed I believe to be the case. These difficulties may be wholly imaginary. If, which seems on other grounds most probable, P is not even approximately a scalar when the body is strained, we should not be able to deduce that $\mathbf{SC}\Theta$ was even approximately zero. In this case, the adjustments brought about in a steady field of the kind just contemplated by reason of the equations (25), would probably be mainly strain adjustments that would not cause $\mathbf{C}$ and Θ to vary much, if at all, from parallelism. These strains, of course, might very well have hitherto escaped detection.

These difficulties are, however, sufficiently serious to make it necessary to consider the results of assuming P independent of the strain. The most important of these results are easily seen to be (1) that, although the connections between the PELTIER effect and the electromotive forces in a circuit of different metals whose junctions are at different temperatures would on the theory of reversibility be the same as is usually supposed, yet there would on that theory, taken alone, be no THOMSON effect [equation (19)], and (2) that there would be no thermoelectric effects in a circuit of a single metal whose various parts were variously stressed. These two, then, would have to be explained on the theory of irreversibility, and no quantitative connection need be expected between the THOMSON effect and the main thermoelectric effects.

92. These difficulties seem to me not to be confined to the particular form of theory developed in this paper. For instance, there seems as much reason to suppose Professor J. J. THOMSON'S σ_x, σ_y, σ_z ('Applications,' 1st ed., § 53) to be independent of the strain as the present P. And I may remark in passing that similar statements may be made with regard to the C′ (*ibid.*, § 43) introduced to explain the HALL effect. [By § 84 above, it is obvious that on the present theory it is impossible to explain the HALL effect by such a term owing to the results other than the HALL effect that would ensue from the term.]

93. Our chief conclusions, so far, may be thus summarised:—

(1.) *If* P *be assumed to be dependent upon* Ψ, *the theory of reversibility suffices to explain all the known experimental facts of thermoelectricity.*

(2.) *If, as there is some reason to believe,* P *be independent of* Ψ, *the main thermoelectric effects must be explained on the theory of reversibility, but the* THOMSON *effect and the thermoelectric effects observed in a circuit of a single metal differently stressed in different parts must be explained on the theory of irreversibility. In this case there is no such connection between the* THOMSON *effect and other thermoelectric effects as is usually supposed.*

(3.) *On the present theory it is impossible to explain thermomagnetic phenomena by the theory of reversibility.*

94. There is little to be said with regard to the thermomagnetic phenomena themselves, as our knowledge of them is almost confined to what is expressed by equations (2) and (9A). It is necessary to remark, however, that the C of equation (2) may be—probably is—not the main cause of what has been described in § 79 above as the longitudinal effect, since an apparent longitudinal effect would be caused (§ 88 above) by any interference with the lines of flow of electricity, and by the variation in the resistance due to any cause. More than one effect of these kinds will be noticed below.

But, of course, the existence of B and C may involve results of a kind other than thermomagnetic, which are practically measurable. Besides equations (2) and (9A), B and C occur in equations (13A) to (18A). Equations (13A) (14A) do not require notice, since in the present state of accuracy of experimental knowledge of thermoelectric quantities the influence of B and C in these equations is negligible. With regard to equations (16A) to (18A), we can trace approximately the effects of B and C in one important class of cases.

95. Suppose we have a plate of uniform thickness (small) in which a current is flowing placed in a strong uniform magnetic field. On account of the current, of course, the uniformity will be disturbed, but only slightly if the strength is great enough. Outside the plate (except in certain conducting wires) we assume that there is no current. Hence by equation (18A) we see that at every point of the boundary of the plate

$$\mathrm{SU}\nu\,(\mathrm{VBK}\Theta - 2\mathrm{CHSK}\Theta) = 0 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (26).$$

Equation (17A) gives by equation (4) § 5 for any portion of the plate

$$\iint \mathrm{S}d\Sigma\,(\mathrm{VBK}\Theta - 2\mathrm{CHSK}\Theta) = 0 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (27).$$

This is the form in which can be most easily discussed the effect of equation (17A). Let the region to which equation (27) refers be taken as a cylinder, one face of the cylinder being in one face of the plate, and the parallel face somewhere inside the plate. Since the plate is thin, we may suppose the faces of the cylinder large compared with the curved surface, and may, therefore, neglect the portion contributed to the integral of equation (27) by the curved surface compared with the rest of the integral. Now by equation (26) the part of the integral contributed by the face of the plate is zero. Hence putting i for $U\nu$ we have, *approximately*, at any point of the plate

$$Si(V\mathbf{BK}\Theta - 2\mathbf{C}\mathbf{H}S\mathbf{K}\Theta) = 0 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (28).$$

Assuming B and C to be scalars, this may be put in the form

$$Si\mathbf{K}^{-1}\Theta = 2CB^{-1}Si\mathbf{H}S\mathbf{K}^{-1}\Theta \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (29).$$

Now assuming, which will be approximately—*exactly* at the boundary—true, that i is perpendicular to $\mathbf{K}$, we have

$$\Theta = -iSi\Theta + \mathbf{K}S\mathbf{K}^{-1}\Theta + i\mathbf{K}Si\mathbf{K}^{-1}\Theta.$$

Hence by equation (2), § 81,

$$\varpi(\Theta + iSi\Theta) = (A - C\mathbf{H}^2)(\mathbf{K}S\mathbf{K}^{-1}\Theta + i\mathbf{K}Si\mathbf{K}^{-1}\Theta)$$
$$+ B(V\mathbf{KH}S\mathbf{K}^{-1}\Theta + Vi\mathbf{KH}Si\mathbf{K}^{-1}\Theta).$$

Let us split this vector up into its components parallel to the three vectors $\mathbf{K}$, i, and $V\mathbf{KH}$. For this purpose notice that since $\mathbf{K}$ is perpendicular to i,

$$Vi\mathbf{KH} = -\mathbf{K}Si\mathbf{H} + iS\mathbf{KH}$$
$$i\mathbf{K} = (V\mathbf{KH} + iSi\mathbf{KH})/Si\mathbf{H}.$$

Thus

$$\varpi(\Theta + iSi\Theta) = \mathbf{K}\{(A - C\mathbf{H}^2)S\mathbf{K}^{-1}\Theta - BSi\mathbf{K}^{-1}\Theta Si\mathbf{H}\}$$
$$+ iSi\mathbf{K}^{-1}\Theta\{(A - C\mathbf{H}^2)Si\mathbf{KH}/Si\mathbf{H} + BS\mathbf{KH}\}$$
$$+ V\mathbf{KH}\{(A - C\mathbf{H}^2)Si\mathbf{K}^{-1}\Theta/Si\mathbf{H} + BS\mathbf{K}^{-1}\Theta\} \quad . \quad . \quad (30)$$

from which, by substituting for $Si\mathbf{K}^{-1}\Theta$ from equation (29), we obtain

$$\varpi(\Theta + iSi\Theta) = B^{-1}S\mathbf{K}^{-1}\Theta[\mathbf{K}B\{A - C(\mathbf{H}^2 + 2S^2i\mathbf{H})\}$$
$$+ i2C\{(A - C\mathbf{H}^2)Si\mathbf{KH} + BS\mathbf{KH}Si\mathbf{H}\}$$
$$+ V\mathbf{KH}\{2C(A - C\mathbf{H}^2) + B^2\}] \quad . \quad . \quad . \quad . \quad . \quad . \quad (31).$$

96. This transformation is not likely to give us clearer ideas of what takes place when a stream of heat is made to flow in the plate which is large compared with the

streams due to ordinary electric resistance and thermoelectric phenomena. The original form of ϖ is more suitable for that purpose. We assume then that the only heat effects are due to purely electrical causes. In this case, if the faces of the plate are thermally similar, we may assume that $Si\Theta$ has opposite values at points situated symmetrically on opposite sides of the plane midway between the faces of the plate. The effect of $\varpi i Si\Theta$ will be then merely to make the current stronger or weaker in the middle of the plate than near the faces, and, therefore, to increase the apparent resistance of the plate. We have already seen that unless P (assumed invariably to be a scalar in this connection) be independent of the strain, $S\mathbf{K}^{-1}\Theta$ is zero, so that the whole expression on the right of equation (31) is zero. If, however, P be independent of the strain, the term in $V\mathbf{KH}$ in equation (31) would indicate a HALL effect. The presence of $S\mathbf{K}^{-1}\Theta$ in this term, however, serves to show that probably this is not the true explanation of the HALL effect.

The HALL effect may be explained by saying that there is an electromotive force $hV\mathbf{KH}$, where h may be called the coefficient of the HALL effect. It has been found experimentally that this coefficient is by no means independent of $T\mathbf{H}$—that, in fact, in certain cases it changes sign when a definite strength of magnetic field is reached. The above work indicates how, on the present theory, we may seek to explain such an effect. For this purpose it must be remembered that equations (30) and (31) are only true if g is the only term in x which involves $\mathbf{H}$.

97. Let us now assume that the electric resistance is a function of $\mathbf{H}$, and let us incorporate in g the term of x thus depending upon $\mathbf{H}$. We must add a term $-S\mathbf{K}r\mathbf{K}/2$ to the former value of g; r being a function (of Class II. of § 9) of ϑ, Ψ and $\mathbf{H}$. For the sake of definiteness give r the form $-bS\mathbf{H}c\mathbf{H}$, where b, c are functions of the same classes as B and C respectively. Thus, as can be easily seen, to get r' or r'' from r we have merely to change b, c and $\mathbf{H}$ into b', c', and $\mathbf{H}'$ or b'', c'', and $\mathbf{H}''$ respectively. Equations (8A), (9A), (13A), and (16A) must now be changed to

$$g = -S\mathbf{K}\varpi\Theta + S\mathbf{K}b\mathbf{K}S\mathbf{H}c\mathbf{H}/2 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (32),$$

$$\mathbf{E}_g = -\varpi\Theta + b\mathbf{K}S\mathbf{H}c\mathbf{H} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (33),$$

$$\theta\dot{f}_g = -S\mathbf{K}\left(2\,[0] + 3\,[1] + 4\,[2]\right)\Theta + 2S\mathbf{K}b\mathbf{K}S\mathbf{H}c\mathbf{H} - \theta S\mathbf{K}_1\varpi_1\nabla_1 \quad . \quad . \quad (34),$$

$$\mathfrak{b}_g/4\pi = V\mathbf{BK}\Theta - 2\mathbf{C}\mathbf{H}S\mathbf{K}\Theta - c\mathbf{H}S\mathbf{K}b\mathbf{K} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (35).$$

Hence, in place of equation (28) we now have

$$Si\left(V\mathbf{BK}\Theta - 2\mathbf{CH}S\mathbf{K}\Theta - c\mathbf{H}S\mathbf{K}b\mathbf{K}\right) = 0 \quad . \quad . \quad . \quad . \quad . \quad . \quad (36).$$

Assuming B, C, b, c, to be all scalars, and putting, as is permissible in this case, $b = 1$, instead of equation (29) we have

$$SiK^{-1}\Theta = B^{-1}SiH.(2CSK^{-1}\Theta + c) \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (37),$$

and in place of (31),

$$\begin{aligned} E_g - \varpi iSi\Theta = -B^{-1}[KB\{SK^{-1}\Theta[A - C(H^2 + 2S^2iH)] - c(H^2 + S^2iH)\} \\ + i(2CSK^{-1}\Theta + c)\{(A - CH^2)SiKH + BSKHSiH\} \\ + VKH\{(2CSK^{-1}\Theta + c)(A - CH^2) + B^2SK^{-1}\Theta\}] \quad . \quad . \quad . \quad . \quad (38)\end{aligned}$$

which simplifies when $SK^{-1}\Theta$ is zero (which it certainly approximately is in the experiments made to determine the coefficient of the HALL effect, whether P be independent of Ψ or not) to

$$\begin{aligned} E_g - \varpi iSi\Theta = B^{-1}c[KB(H^2 + S^2iH) \\ - i\{(A - CH^2)SiKH + BSKHSiH\} \\ - VKH(A - CH^2)] \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (39).\end{aligned}$$

Owing to the term in i on the right as well as the term in i on the left, there may be an apparent increase of resistance. The term in **K** shows that there will also be an increase $-c(H^2 + S^2iH)$ in the resistance. The term in **VKH** shows that the present assumptions lead to a HALL effect, whose coefficient $= -B^{-1}c(A - CH^2)$.

With regard to the new term $2SKbKSHcH$ in equation (34), it should be noticed that since it is quadratic in **K**, it would have no influence on the apparent THOMSON effect, but only upon the apparent resistance as measured by heat effects.

That we can explain the HALL effect on the present theory is of some interest, because, as remarked in § 92 above, we cannot explain it on the present theory in the ordinary way. Nor can we hope to explain it by the term $-{}_K\nabla x$ in **E** [equations (29), § 50, and (20), § 35] for $V_K\nabla VKH = V\zeta V\zeta H = -2_H$, whereas $V_K\nabla\ ({}_K\nabla x) = 0$. It should be noticed that the difficulties in the way of explaining the HALL effect by a term $-C'SCDH/2$ do not apply to explaining the magnetic rotation of the plane of polarised light, since this is equally well explained by substituting **d** for **D**.

98. One effect of g still remains for consideration. It is necessary to consider it if only to show that it leads to no results large enough to be experimentally tested. It also helps to show how the various interferences with the lines of flow, several times mentioned above, are mainly brought about.

In writing down equation (9A) it was mentioned that E_g did not contain the part of **a** due to g. We have indirectly taken account of the effect of a_g in part, by the considerations just given in leading up to the explanation of the HALL effect. We have not, however, thereby taken full account of a_g. To do this in the manner illustrated above, we should require to study the effect b_g had in modifying the whole electromotive force instead of the part E_g. We proceed then, to a more general examination of the effect of the term $-a_g$ in **E**.

We will now drop the suffix g from **a** and **b**, since we shall not suppose x to contain

$\mathbf{H}$ except by reason of the term g, so that $\mathbf{b}_g$ of equation (35) now stands for the full value of $\mathbf{b}$. $S\nabla\mathbf{a}$ and $[SU\nu\mathbf{a}]_{a+b}$ are arbitrary. Let us, since it does not affect any physically measurable quantity, assume them both to be zero. Since $\mathbf{b} = V\nabla\mathbf{a}$, $[Vd\Sigma\mathbf{a}]_{a+b} = 0$, if we considered an analogue in which $\mathbf{b}$ stood for an electric current, $4\pi\mathbf{a}$ would be the magnetic force due to the currents in a space containing no magnets. This analogue will serve to give us a very fair general idea of the effect of $-\mathbf{a}$ in $\mathbf{E}$. Since [equation (35)] every term of $\mathbf{b}$ contains $\mathbf{K}$, $\mathbf{b}$ will be confined to conductors where there are conduction currents. Thus, in the same conductor we shall have the "real current" and the "current of the analogue." The current of the analogue may be disposed with reference to the real current in one of three ways. It may be mainly parallel to the real current, or it may circulate round the real current mainly at right angles to it, or it may circulate round it spirally. In the first and third cases we see by the analogy that there would be an electromotive force due to $\mathbf{a}$ in the general field approximately parallel everywhere to the part of the magnetic force due to the real current. In the conductor itself, then, the resulting electromotive force would cause the real current to move spirally, and would, therefore, apparently increase the resistance. In the second and third cases we see by the analogy that there would result effects due to the local state of affairs, so that where $\mathbf{b}$ was large there the effect of $\mathbf{a}$ on $\mathbf{E}$ would be large. In a case of this sort we should have to examine further before we could say what the local effect would be.

It is easy to see that in the experiments for determining the quantities connected with the HALL and thermomagnetic effects, the second of the above cases very approximately represents the state of affairs. For, by equation (35), it is only in the plate, where Θ or $\mathbf{H}$ is very large compared with the rest of the circuit, that $\mathbf{b}$ will have a sensible value. Hence there must be a strong local current of the analogue, that is, a current which does *not* go round the circuit parallel to the real current.

It follows that the main physical effects of $\mathbf{a}$ are those that were considered in dealing with the HALL effect.

E. *Contact Electromotive Force.*

99. Our knowledge of this is not very accurate, but, besides the fact that contact-force certainly exists, and that it has been in numerous individual cases measured with fair accuracy, the following seems to stand out with considerable certainty.

If the (apparent) electromotive force from one material, a, to another, b, when they are in contact be denoted by $a \mid b$, then the equation

$$a \mid b + b \mid c + c \mid a = 0 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (1)$$

is true if all three materials are conductors, but is not true if they are not all conductors.

This force cannot apparently be explained by any term in l of the kind we have hitherto supposed l to contain. Suppose, then, l to contain a term* $\mathrm{Sd}a\Delta$ where a is of the same class as A or ϖ [equations (5), § 81.] Thus $a\sigma$ is an intensity, and $a_c\tau$ a flux, a_c being the conjugate of a.

The portion of L contributed by this term is $\iiint \mathrm{Sd}a\Delta d\varsigma$, or [eq. (4), § 5], $\iint \mathrm{Sd}a d\Sigma$. Hence the effects of supposing l to contain a term $\mathrm{Sd}a\Delta$ are identical with those of supposing l_s to contain a term $[\mathrm{Sd}a\mathrm{U}\nu]_{a+b}$.

So far as electric phenomena are concerned it is quite easy to see the effect of this term. In place of equations (31), § 50, and (2), § 57, we shall have

$$\mathbf{E}_s = [v\mathrm{U}\nu]_{a+b} = [a\mathrm{U}\nu]_{a+b} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (2),$$

or, if a be a scalar,

$$[v]_{a-b} = [a]_{a-b} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (3).$$

100. Although this term involves a contact force it does not explain the known facts, since, as can be easily seen, the contact force here obtained is such that equation (1) would be invariably true. We seem then to be driven to the conclusion that to explain contact force, l_s cannot any longer be assumed to be zero.

Adopting the suggestion of Professor CHRYSTAL, 'Encyc. Brit.,' 9th ed., vol. 8, p. 85, we will assume that there is no real contact force between conductors. This simplification is not, of course, necessary on the present theory, but the simpler the assumptions—so long as they are not intrinsically improbable—the better. Professor CHRYSTAL shows that the assumption serves to explain all the known facts, the apparent contact force between conductors being explained by the difference of their contact forces with one and the same dielectric.

We now assume that l_s contains the term† $\mathrm{S}a\mathrm{U}\nu_a[\mathbf{d}]_{a+b}/2$ where a is of the same class as before, but now depends on Ψ_a, Ψ_b and θ, and where of course the suffix a has nothing to do with the linear vector function a. It is assumed that a is zero for a surface of separation of two conductors.

In place of equations (2), (3), we now have

$$\mathbf{E}_s = [v\mathrm{U}\nu]_{a+b} = a\mathrm{U}\nu_a. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (4)$$

$$[v]_{a-b} = a \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (5).$$

* It may be asked, Why make the differentiations act on a as well as $\mathbf{d}$, why not assume the term to be $\mathrm{Sd}_1 a\nabla_1$? The answer is that this leads to a more complicated result, namely (1), to the same contact force as the term chosen, (2), to a term $-a\Delta$ in $\mathbf{E}$, and (3), to a stress which involves the space derivatives of $\mathbf{d}$. It is best to assume, if possible, a term which involves what we know to be true and nothing more.

† Perhaps it would be better, as more general to suppose l_s to contain the term $[\mathrm{S}\alpha\mathrm{U}\nu\,\mathbf{d}]_{a+b}$ where α is of the same class as a, and α_b is not merely characteristic of the substance b, but depends on *both* the substances bounded, *i.e.*, where, in general, $\alpha_{a-b} + \alpha_{b-c} + \alpha_{c-a}$ is not zero. Equations (4), (5), (6) will still be true if we put $a = \alpha_{a-b}$.

It might be thought that equation (4) could not be true in general unless α was a scalar in general. This, however, is not the case. From equation (4) it certainly follows that $\mathrm{VU}\nu\alpha\mathrm{U}\nu = 0$, but this, by virtue of the dependence of α on strain is merely an equation of condition satisfied by the strain. The equation $[v\mathrm{U}\nu]_{a+b} = \alpha\mathrm{U}\nu_a$ may indeed be written

$$[v]_{a-b} = -\mathrm{U}\nu\alpha\mathrm{U}\nu \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (6),$$

which is, of course, more general than equation (5), since here α is not assumed a scalar. It may be noted that $\mathrm{V}\, d\Sigma'\alpha'\, d\Sigma' = m\chi\mathrm{V}\, d\Sigma\alpha\, d\Sigma$, so that the two conditions, $\mathrm{VU}\nu'\alpha'\mathrm{U}\nu' = 0$ and $\mathrm{VU}\nu\alpha\mathrm{U}\nu = 0$, are identical.

101. It should be remarked here that on the present theory the term just introduced would have no thermal effect in steady fields, and, therefore, no connection with the PELTIER effect. (See § 85 above.)

We have been obliged to suppose l_s no longer zero. Before discussing the modifications this entails in the general results above, the last application in the present paper of those results will be made.

This is the place where it would be proper to discuss electrolysis in connection with the present theory. This I do not propose to do, because the mathematical machinery of this paper would require some important modifications to enable us to deal with such subjects as diffusion, the motion of the ions, &c., and because the subject is a very large one, and would, perhaps, unduly extend the length of the present paper.

F. *The Transference of Energy through the Field.*

102. On the present theory, in which the principle enunciated in equations (24), (25), § 36, required strong confirmation, it was necessary to show that it agreed in every particular with the generally accepted views as to frictional forces being derivable from a dissipation function in Lord RAYLEIGH's sense, and also with the much more certainly established truths treated of in the theory of conduction of heat. The only way to establish this last seemed to be to show that as a result of the principle there was a time flux of intrinsic energy, one term of which was what, in the theory of conduction of heat, is called the time flux of heat. This led to the necessity of finding the time flux of intrinsic energy in general. We are thus brought on to ground which has hitherto been regarded as belonging exclusively to Professor POYNTING—the transference of energy through the field.

103. Let us now examine how far the results of the present theory are consistent with those of Professor POYNTING. Let $\mathbf{L}$ be a flux such that

$$\dot{\mathrm{E}} - \mathrm{P}_e = \iint_b \mathrm{S}\mathbf{L}\, d\Sigma \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (1),$$

so that **L** may be called the time flux of intrinsic energy. By Proposition VIII., § 10,

$$(\phi' + \Phi_f')\, d\Sigma' = \chi\,(\phi + \Phi_f)\, d\Sigma.$$

Hence we see by equation (25), § 49, that

$$\mathbf{L} = -(\phi + \Phi_f)\,\chi'\dot{\rho}' + v\mathbf{C} - \mathrm{V}\,(\dot{\mathbf{A}} + \mathbf{a})\,\mathbf{H}/4\pi - (\dot{\theta}_{\circledcirc}\nabla\lambda + \theta_{\circledcirc}\nabla x) \quad . \quad . \quad (2).$$

Now, ('Phil. Trans.,' 1884, Part II., pp. 343 to 349) Professor POYNTING'S result expressed in similar notation would be (calling the time flux of energy **P**)

$$\mathbf{P} = -(\phi + \Phi_f)\,\chi'\dot{\rho}' - \mathrm{V}\,\{\nabla v + (\dot{\mathbf{A}} + \mathbf{a})\}\,\mathbf{H}/4\pi - (\dot{\theta}_{\circledcirc}\nabla\lambda + \theta_{\circledcirc}\nabla x). \quad . \quad (3).$$

It is scarcely necessary to remark that we have here generalised his expression* by the insertion of the terms $-(\phi + \Phi_f)\,\chi'\dot{\rho}' - \mathrm{V}\mathbf{a}\mathbf{H}/4\pi - (\dot{\theta}_{\circledcirc}\nabla\lambda + \theta_{\circledcirc}\nabla x)$ and have substituted for his **E** what he means by it, namely, $-(\dot{\mathbf{A}} + \nabla v)$. It might be thought at first that this is not quite what he means by **E** since he incorporates in it terms depending on the motion of the body with reference to the lines of magnetic induction. Remembering, however, that equation (6), § 60, and the equation $\mathbf{E}_0 = -(\dot{\mathbf{A}} + \nabla v)$ are identical, it will be seen that these terms have been here incorporated.

104. The *direct* interpretation of equations (2) and (3) is, of course, widely different. Let us see if they have the same physical significance, that is, whether they lead to the same rate of increase of intrinsic energy in any finite space.

For this purpose it must be asked whether or not $\iint_b \mathrm{S}\,(\mathbf{L} - \mathbf{P})\, d\Sigma$ is zero. Now

$$4\pi\,(\mathbf{L} - \mathbf{P}) = 4\pi v\mathbf{C} + \mathrm{V}\nabla v\mathbf{H} = \mathrm{V}\nabla\,(v\mathbf{H})\, . \quad . \quad . \quad . \quad . \quad . \quad (4).$$

Hence $4\pi\iint_b \mathrm{S}\,(\mathbf{L} - \mathbf{P})\, d\Sigma = \iint_b \mathrm{S}\, d\Sigma\nabla\,(v\mathbf{H})$, or by equation (3), § 5,

$$4\pi\iint_b \mathrm{S}\,(\mathbf{L} - \mathbf{P})\, d\Sigma = \int_b v\mathrm{S}\, d\rho\mathbf{H} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (5),$$

where the line integral† is to be taken over all lines of discontinuity on the true

* I only say—generalised his *expression*—since *some such* terms as have been added in the text would, on Professor POYNTING'S own theory, be included in the vector **L**, defined by equation (1), as the time flux of *intrinsic* energy. The result of the present paper is, however, in all strictness much more general than his, since it has not among other things been assumed that all the bodies in space are isotropic with reference to specific inductive capacity, resistance, and magnetic permeability.

† It may be well to notice that by the conventions of § 5, above, if the closed curve be regarded as bounding, not the regions of the true boundary, but the part of the surface of discontinuity in the region of space under consideration, the sign of the line integral must be changed.

boundary of the region considered, that is, over the trace on that surface of surfaces of discontinuity. The element $d\rho$ is, of course, taken twice, namely, once for each of the two regions of the true boundary which it bounds. Since $d\rho$ is in the surface of discontinuity, and the component of **H** parallel to that surface is not discontinuous, we see that $[\mathrm{S}d\rho\mathbf{H}]_{a+b} = 0$. Hence the part contributed by $d\rho$ to the line integral may be written $[v]_{a-b}[\mathrm{S}d\rho\mathbf{H}]_a$ or

$$v_{a-b}\mathrm{S}d\rho_a\mathbf{H} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (6).$$

If then v is continuous, the line integral is zero. It has already appeared [equation (2), § 57] that if l_s is everywhere zero this is the case. Hence, unless l_s exist, the physical results of supposing **P** to be the time flux of intrinsic energy are identical with those of supposing **L**.

If l_s exist, we have at present no right to say that on the present theory **L** may be taken as the time flux; but in § 111, below, this will be proved. The conclusion is then that, to explain the rate of variation of energy, Professor POYNTING's flux **P** must be supplemented by a finite flux $\mathbf{P}_s$ along surfaces of discontinuity in the potential, where

$$4\pi\mathbf{P}_s = [v\mathrm{VU}\nu\mathbf{H}]_{a+b} = v_{a-b}\mathrm{VU}\nu_a\mathbf{H} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (7).$$

[In verifying the sign of this expression attention must be paid to the caution in the last foot-note.] This of course is rather an unnatural, though by no means absurd, result, and therefore I think it better to regard **L** as, more probably than **P**, representing the true time flux of intrinsic energy. Another reason for preferring **L** to **P** is that for a field at rest, *i.e.*, such that $\dot{\rho}'$, $\dot{\theta}$, Θ, **C** and **c** are everywhere zero, **L** is zero, whereas $\mathbf{P} = \mathrm{V}\mathbf{H}\nabla v/4\pi$.

In now contrasting Professor POYNTING's result with that of the present paper, we will suppose v continuous.

105. Very shortly after the first publication of Professor POYNTING's paper, Professor J. J. THOMSON in criticising it remarked ('B. A. Reports,' 1885, p. 151). "This [Professor POYNTING's] interpretation of the expression for the variation in the energy seems open to question. In the first place it would seem impossible, *à priori*, to determine the way in which the energy flows from one part of the field to another by merely differentiating a general expression for the energy in any region, with respect to the time, without having any knowledge of the mechanism which produces the phenomena which occur in the electromagnetic field; for although we can, by means of HAMILTON's or LAGRANGE's equations, deduce from the expression for the energy the forces present in any dynamical system, and therefore the way in which the energy will move, yet for this purpose we require the energy to be expressed in terms of the coordinates fixing the system, and it will not do to take any expression which happens to be equal to it. The problem of finding the way in which the energy is transmitted in a system whose mechanism is unknown, seems to be an

indeterminate one ; thus, for example, if the energy inside a closed surface remains constant we cannot, unless we know the mechanism of the system, tell whether this is because there is no flow of energy either into or out of the surface, or because as much flows in as flows out. The reason for this difference between what we should expect and the result obtained in this paper is not far to seek." He then goes on to point out* how, so far from **P** being necessarily the time flux of energy, $\mathbf{P} + \mathrm{V}\nabla\epsilon$ where ϵ is any vector, such that at surfaces of discontinuity $[\mathrm{VU}\nu\epsilon]_{a+b} = 0$, might equally well be taken as the time flux of energy. It so happens that (assuming v continuous) $\mathbf{L} - \mathbf{P}$ is such a vector, so that the difference between the result arrived at in this paper and Professor POYNTING's is just such a case as Professor THOMSON warned us to expect.

We cannot then say that either **L** or **P** is the time flux of energy, but only that if we assume either the one or the other (**P** being supplemented with $\mathbf{P}_s$) to be the flux, the real changes of intrinsic energy will be accounted for.

106. Notwithstanding Professor THOMSON's warning, many subsequent writers seem to have taken Professor POYNTING's theories for established facts. The following statement of Professor POYNTING especially seems to have grown to be accepted almost universally as a commonplace truth ['Phil. Trans.,' 1884, Part II., p. 361]:— "I think it is necessary that we should realise thoroughly, that if we accept MAXWELL's theory of energy residing in the medium, we must no longer consider a current as something conveying energy along the conductor. A current in a conductor is rather to be regarded as consisting essentially of a convergence of electric and magnetic energy from the medium upon the conductor, and its transformation there into other forms." Now, if we take **L** as the true time flux of energy, we see that one way in which we *must* regard a current is precisely the way Professor POYNTING denies us, namely, "as something conveying energy along the conductor." In fact, from the term $v\mathbf{C}$ in **L**, we see that in this respect, as in so many others, a current and the potential are the exact analogue of a liquid current and its pressure. Without doubt, the view that **L** is the true flux is simpler for steady fields than the view that **P** is. This statement is not so obvious—perhaps on the whole not true—for varying fields.

It is easy to contrast in detail the two views in all the particular cases Professor POYNTING considers. This may, therefore, be omitted here.

G. *The General Effects of the Existence of l_s.*

107. The general equations above established must now be modified on account of l_s. In considering electrolysis on the present theory it would be necessary to suppose l_s to contain **D** or **C** or both, as well as **d**. For the sake of simplicity we shall not

* This is not put quite in the form Professor THOMSON puts the case.

make this supposition. l_s will be assumed a function, then, of $\mathbf{d}_a$, $\mathbf{d}_b$, θ, Ψ_a, Ψ_b. As throughout the rest of this paper, we assume that there is no slipping at the surface. This leads to a relation between Ψ_a and Ψ_b. Let i, j, k as usual stand for a set of mutually perpendicular unit vectors, which are, however, functions of the position of a point. Let

$$i = \mathrm{U}\nu_a, \quad i' = \mathrm{U}\nu_a' \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (1);$$

j and k are thus parallel to the surface.

We have

$$\Psi\omega = \Sigma i\mathrm{S}i\Psi i\mathrm{S}i\omega + \Sigma\,(j\mathrm{S}k\omega + k\mathrm{S}j\omega)\,\mathrm{S}j\Psi k$$

$$= \Sigma i\mathrm{S}i\omega\,(\chi i)^2 + \Sigma\,(j\mathrm{S}k\omega + k\mathrm{S}j\omega)\,\mathrm{S}\chi j\chi k,$$

where the summation sign implies that i, j, k are to be changed cyclically. Since there is no slipping the strains in the surface of each region bounded are the same or

$$\chi_a j = \chi_b j, \qquad \chi_a k = \chi_b k$$

Hence, putting

$$2\overline{\Psi} = \Psi_{a+b}, \qquad 2\Gamma = \Psi_{a-b}i = [\Psi\mathrm{U}\nu]_{a+b} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (2),$$

it at once follows that

$$\Psi_{a-b}\omega = -\,2\Gamma\mathrm{S}i\omega + 2i\,(\mathrm{S}j\Gamma\mathrm{S}j\omega + \mathrm{S}k\Gamma\mathrm{S}k\omega) \quad . \quad . \quad . \quad . \quad . \quad . \quad (3).$$

Thus Ψ_a and Ψ_b can be expressed in terms of the independent variables $\overline{\Psi}$ and Γ. That these last are independent is seen thus. The deformation in the neighbourhood of a point on the bounding surface requires for complete specification a knowledge of the following three things: (1) the pure strain of an element of the surface, (2) the displacement of a point in the region a near to the element of surface relative to the latter when purely strained, (3) a similar displacement in the region b. These three are independent, and each requires three scalars to specify it—nine in all, the same number as is required to specify $\overline{\Psi}$ and Γ. Thus l_s is a function of the variables $\mathbf{d}_a$, $\mathbf{d}_b$, $\overline{\Psi}$ and Γ.

108. The part of δl_s depending on $\delta\Psi_a$ and $\delta\Psi_b$ is $-\,\mathrm{S}\delta\overline{\Psi}\zeta\overline{\mathrm{C}}l_s\zeta - \mathrm{S}\delta\Gamma_\Gamma\nabla l_s$* where $\overline{\mathrm{C}}$ stands for ${}_{\overline{\Psi}}\mathrm{C}$. Put now

* We may dispense with Γ altogether thus. Put $\overline{\Psi} + \mathrm{V}\Gamma(\;\;) = \Pi$. Then with the meaning of ${}_\Pi\mathrm{C}$ explained on page 103 of former paper,

$${}_\Gamma\nabla = \mathrm{V}\zeta_\Pi\mathrm{C}\zeta, \; \overline{\mathrm{C}} = {}_\Pi\mathrm{C} - \mathrm{V}_\Gamma\nabla(\;\;)/2,$$

i.e., $\overline{\mathrm{C}}$ is the pure part, and ${}_\Gamma\nabla/2$ the rotation-vector of ${}_\Pi\mathrm{C}$.

$$m_s = ds'/ds = \mathrm{T}d\Sigma'/\mathrm{T}d\Sigma = m\mathrm{T}\chi'^{-1}i \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (4).$$

$$\left.\begin{array}{c}\{\phi_s'\} = -m_s^{-1}\{\chi \mho l_s \chi' - \tfrac{1}{2}[\chi_{\mathrm{r}}\nabla l_s \mathrm{S}(\ \ )\chi \mathrm{U}\nu + \chi\mathrm{U}\nu\mathrm{S}(\ \ )_{\mathrm{s}}^{\mathrm{3}}\chi_{\mathrm{r}}\nabla l_s]\} \\ \phi_s' = [\{\phi_s'\}]_{a+b}\end{array}\right\} \quad . \quad . \quad (5).$$

Then it is easy to show, after the manner of § 39, that

$$\delta l_s = m_s[\mathrm{S}\delta\rho_1'\{\phi_s'\}\nabla_1']_{a+b} - \mathrm{S}\delta\mathrm{d}[{}_{\mathrm{d}}\nabla l_s]_{a+b} \quad . \quad . \quad . \quad . \quad . \quad . \quad (6).$$

To see what modifications must be made in equation (9), § 45, consider first the first term on the right of equation (6). This contributes to $\delta\mathrm{L}$ for a finite portion of space $\iint \mathrm{S}\delta\rho_1'\{\phi_s'\}\nabla_1'ds'$. Put in this $\nabla_1' = -i'\mathrm{S}i'\nabla_1' - i'\mathrm{V}i'\nabla_1'$. Thus

$$\iint \mathrm{S}\delta\rho_1'\{\phi_s'\}\nabla_1'ds' = -\iint \mathrm{S}i'\nabla_1'\mathrm{S}\delta\rho_1'\{\phi_s'\}i'ds' - \iint \mathrm{S}\delta\rho_1'\{\phi_s'\}(i'\mathrm{V}i'\nabla_1')\,ds'.$$

109. We will anticipate somewhat here, as the effect is a considerable simplification. The first of the terms on the right involves the vector $-\mathrm{S}i'\nabla'.\delta\rho'$, and this is the only term involving space derivatives of $\delta\rho'$ that cannot be transformed into terms involving $\delta\rho'$ only. Now the vector coefficient of this vector, like that of $\delta\rho'$ in $\delta\mathrm{L} + \Sigma\mathrm{Q}\delta q$, must be zero. For $-\mathrm{S}i'\nabla'.\delta\rho'$ is the normal space rate of variation of $\delta\rho'$, and it is clear that we can *at the surface* change this arbitrarily without changing $\delta\rho'$ at any point of space. [This cannot be said of the tangential space rates of variation of $\delta\rho'$, for a change in these causes a change of the same order of magnitude in $\delta\rho'$ at all points of the surface.] Hence we obtain the equation

$$\phi_s'i' = 0 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (7),$$

or by equation (5), since $i' = mm_s^{-1}\chi'^{-1}i$,

$$\left.\begin{array}{l}[m\chi\{\mho l_s\mathrm{U}\nu + \tfrac{1}{2}({}_{\mathrm{r}}\nabla l_s - \mathrm{U}\nu\mathrm{S}\mathrm{U}\nu_{\mathrm{r}}\nabla l_s)\}]_{a-b} \\ \quad = [m\chi]_{a+b}\mho l_s i + \tfrac{1}{2}[m\chi]_{a-b}({}_{\mathrm{r}}\nabla l_s - i\mathrm{S}i_{\mathrm{r}}\nabla l_s) = 0\end{array}\right\} \quad . \quad . \quad . \quad . \quad (8).$$

The geometrical meaning of equation (7) should be noticed. It reduces the six coordinates of the self-conjugate ϕ_s' to three. ϕ_s' operating on any vector reduces it to the tangent plane. It may be said to act only on vectors in the plane and to strain them in the plane.

110. We now have

$$\iint \mathrm{S}\delta\rho_1'\{\phi_s'\}\nabla_1'\,ds' = -\iint \mathrm{S}\delta\rho_1'\phi_s'(i'\mathrm{V}i'\nabla_1')\,ds',$$

ds' on the left being taken, as usual, twice, but on the right only once. That we may substitute $S\delta\rho_1'\phi_s'(i'Vi'\nabla_1')$, or $S\delta\rho_1'[\{\phi_s'\}]_{a+b}(i'Vi'\nabla_1')$ for $[S\delta\rho_1'\{\phi_s'\}(i'Vi'\nabla_1')]_{a+b}$ is obvious, from the fact that the space derivatives of $\delta\rho'$ involved are only the tangential ones, which are the same for both regions bounded, because there is no slipping. The boundary of the surface in question, like the boundary of any volume, must be supposed to involve not only the geometrical boundary, but also any lines of discontinuity on it. With this meaning for the boundary, we have, by equation (3), § 5,

$$\iint S d\rho_1'\{\phi_s'\}\nabla_1' ds' = -\int S\delta\rho'\phi_s'(i'd\rho') + \iint S\delta\rho'\phi_{s1}'(i_1'Vi'\nabla_1')\,ds' \quad . \quad (9).$$

The geometrical meaning of $i_1'Vi'\nabla_1'$ should be noticed. By equation (3), § 5, we have

$$\iint i_1'Vi'\nabla_1'\,ds' = \int i'd\rho' \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (10),$$

from which, by limiting the portion of surface considered to an element bounded by lines of curvature, it can easily be deduced that

$$i_1'Vi'\nabla_1' = i'(r^{-1} + r'^{-1}) \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (11),$$

where r, r' are the principal radii of curvature, reckoned positive or negative, according as the centre of the corresponding curvature is in the region a or b $[i' = U\nu_a']$. Thus $-i_1'Vi'\nabla_1'$, or $Vi'\nabla'.i'$ may be called the vector curvature of the surface at the point.*

Since $\phi_s i' = 0$, we see by equation (11) that equation (9) may be written

$$\iint S\delta\rho_1'\{\phi_s'\}\nabla_1' ds' = -\int S\delta\rho'\phi_s'(i'd\rho') + \iint S\delta\rho'\phi_{s1}'(i'Vi'\nabla_1')\,ds' \quad . \quad (12),$$

ough this last simplification is not needed for our purposes.

111. We are now in a position to see what alterations l_s occasions in the various equations given above. This may be done in the following semi-tabular form.†

Add to right of 45 (9)

$$-\iint S\delta\mathrm{d}\,[_{\mathrm{d}}\nabla l_s]_{a+b}\,ds + \iint S\delta\rho'\phi_{s1}'(i'Vi'\nabla_1')\,ds' - \int S\delta\rho'\phi_s'(i'd\rho') \quad . \quad . \quad (13).$$

* Using, for the moment, the notation of TAIT'S 'Quaternions,' 3rd edition, §§ 296, *et seq.*, for ρ and dashes, it seems to me that lucidity would be gained by calling ρ'' the vector curvature of the curve at the point considered. Thus the vector curvature of a curve is a vector whose tensor is the ordinary scalar curvature, and which is drawn from the point on the curve in question towards the centre of curvature. By analogy, I would call the vector, drawn from a point on a surface towards the concave side, and equal in magnitude to the sum of the principal curvatures, the vector curvature of the surface at the point.

† In what follows " 45 (9) " stands for " § 45, equation (9)."

Add to 3rd term of 46 (10)

$$\iint (l_s + \lambda_s)\, ds \quad . \quad . \quad . \quad . \quad . \quad (14).$$

46 (11) *Unaltered* . . . . (15).

In place of 46 (12)

$$\lambda_s = -l_s, \quad t_s = 0 \quad . \quad . \quad . \quad . \quad . \quad (16).$$

Add to right of 46 (13)

$$\iint (-\dot{\theta}\partial l_s/\partial\theta + \mathrm{S}\dot{\mathrm{d}}_{\mathrm{d}}\nabla l_s)\, ds - \iint \mathrm{S}\dot{\rho}'\phi_{s1}'(i'\nabla i'\nabla_1')\, ds' + \int \mathrm{S}\dot{\rho}'\phi_s'(i'd\rho') \quad . \quad (17).$$

Add to right of 46 (14)

$$\iint \mathrm{S}\dot{\mathrm{d}}_{\mathrm{d}}\nabla l_s\, ds - \iint \mathrm{S}\dot{\rho}'\phi_{s1}'(i'\nabla i'\nabla_1')\, ds' + \int \mathrm{S}\dot{\rho}'\phi_s'(i'd\rho') \quad . \quad . \quad . \quad . \quad (18).$$

47 (15), 49 (17), 50 (26) *Unaltered* . . . . (19).

Add to right of 49 (18) and 50 (27)

$$-\phi_{s1}'(i'\nabla i'\nabla_1') \quad . \quad . \quad . \quad . \quad . \quad (20).$$

Besides the equation $\phi_s' i' = 0$, we here have

$$[\phi_s'(i'd\rho')]_{e+f+} = 0 \quad . \quad . \quad . \quad . \quad . \quad (21),$$

the suffix $e + f +$ indicating two or more superficial regions bounded by the curve.

49 (19), 49 (20), 50 (28), 50 (29) *Unaltered*. . . . . (22).

In place of 49 (21) and 50 (30)

$$\mathrm{e}_s = [{}_{\mathrm{d}}\nabla l_s]_{a+b} \quad . \quad . \quad . \quad . \quad . \quad (23).$$

49 (22), 50 (31) *Unaltered* . . . . (24).

49 (23), (24), (25) *Unaltered* . . . . (25),

[since all the new terms added to 46 (14) are clearly accounted for by P_e. It will be observed that this statement is true of any term of L not involving a velocity, *i.e.*, of any term which merely contributes to the potential energy].

57 (1) *Unaltered* . . . . (26).

In place of 57 (2)

$$[v\mathrm{U}\nu + {}_{\mathrm{d}}\nabla l_s]_{a+b} = 0 \quad \ldots\ldots\ldots \quad (27).^*$$

112. It is well to point out what the exact physical significance of ϕ'_s is. It implies the existence of a membranous stress, *i.e.*, a stress such as a perfectly flexible membrane could exhibit.

To investigate the properties of such a stress in a membrane coincident with the actual surface, let i', j', k' be three mutually perpendicular unit vectors, so that [equation (1), § 107] the two latter are parallel to the surface. Consider an elementary triangle in the surface at the point under consideration, whose vector edges, taken in the positive direction round the triangle, are yj', $-yj' + zk' = d\rho'$ and $-zk'$. Let $y\mathbf{F}_y$, $\mathbf{F}$, and $z\mathbf{F}_z$ be the forces exerted by the rest of the membrane on the triangle across these three faces respectively. Since all other forces on the element are of a higher order of smallness than these three, we have as the equation of motion

$$\begin{aligned}\mathbf{F} &= -y\mathbf{F}_y - z\mathbf{F}_z \\ &= -\mathbf{F}_y\mathrm{S}j'\,d\rho' + \mathbf{F}_z\mathrm{S}k'\,d\rho' \\ &= -\mathbf{F}_y\mathrm{S}k'\,(i'\,d\rho') - \mathbf{F}_z\mathrm{S}j'\,(i'\,d\rho') \\ &= -\Phi_s'\,(i'\,d\rho'),\end{aligned}$$

where Φ_s' is a linear vector function of a vector. This equation does not completely determine Φ_s' since $i'd\rho'$ is not perfectly arbitrary, but confined to a plane. The arbitrary part of Φ_s' having no physical bearing on the problem in hand may be chosen at will. For present purposes it is convenient to define Φ_s' completely by the equation

$$\Phi_s'\omega = -\mathbf{F}_y\mathrm{S}k'\omega - \mathbf{F}_z\mathrm{S}j'\omega,$$

where ω is a perfectly arbitrary vector. This gives

$$\Phi_s'i' = 0 \quad \ldots\ldots\ldots\ldots \quad (28).$$

[This is not always the most convenient way of choosing the arbitrary part of Φ_s' as, for instance, in the study of surface tension.]

Since the membrane is perfectly flexible $\mathbf{F}_y$ and $\mathbf{F}_z$ are parallel to the tangent plane and, therefore, Φ_s' only operates on vectors in the plane, and strains them in the plane. Thus Φ_s' has four disposable scalars.

Calling the side of $d\rho'$ towards which $i'd\rho'$ points the negative side, the above amounts to saying that the force exerted across the element $d\rho'$ *by* the part of the membrane on the positive side *on* the part on the negative side is $-\Phi_s'(id\rho')$. [The direction round any closed curve on the membrane, which is that of positive rotation

* As with equation (6), § 100, this may be put in the form $v_{a-b} = [\mathrm{U}\nu\mathrm{d}\nabla l_s]_{a-b}$

round i' is, of course, considered positive. Thus, for such a closed curve, $i'd\rho'$ points inwards. This is the reason for taking the positive and negative sides as just defined. It also accounts for the sign given to Φ_s', since the latter is thus brought into harmony with the sign of the linear vector function which represents an ordinary stress.] This stress will be called the stress Φ_s'.

113. We now seek the force and couple per unit surface due to the stress Φ_s'. For this purpose, first take a finite portion of the surface. The force exerted by the stress on any portion of the surface is

$$-\int \Phi_s'(i'd\rho') = -\iint \Phi_{s1}'(i_1'\mathrm{V}i'\nabla_1')\,ds'$$

by equation (3), § 5. Hence the force per unit surface due to the stress is

$$-\Phi_{s1}'(i_1'\mathrm{V}i'\nabla_1').$$

Again, the couple for a finite portion of the surface round any arbitrary origin is

$$-\int \mathrm{V}\rho'\Phi_s'(i'd\rho') = -\iint \mathrm{V}\rho'\Phi_{s1}'(i_1'\mathrm{V}i'\nabla_1')\,ds' - \iint \mathrm{V}\zeta\Phi_s'(i'\mathrm{V}i'\zeta)\,ds'.$$

Hence (by the force per unit surface just obtained) the couple per unit surface

$$= -\mathrm{V}\zeta\Phi_s'(i'\mathrm{V}i'\zeta) = \mathrm{V}\zeta\Phi_s'\zeta - \mathrm{V}i'\Phi_s'i' = \mathrm{V}\zeta\Phi_s'\zeta,$$

by equation (28). Assuming, which will be the case if there be no other couple per unit surface, as is certainly true in our case, that there is no such stress couple per unit surface, we see that $\mathrm{V}\zeta\Phi_s'\zeta = 0$ or Φ_s' is self-conjugate. Thus Φ_s' is of exactly the same type as ϕ_s' and has three disposable coordinates only. [It is not necessary to assume this couple zero since the problem may be treated in an exactly similar manner to that of general stress (former paper, p. 106, *et seq.*).]

114. Now suppose Φ_s' is an "external" stress in the actual surface under consideration. The part of $\Sigma Q\delta q$ due to it will be $\iint S\delta\rho'\Phi_{s1}'(i'\mathrm{V}i'\nabla_1')\,ds' - \int S\delta\rho'\Phi_s'(i'd\rho')$, so that the only way in which the expression (13) is affected by these new terms is that ϕ_s' must be changed into $\phi_s' + \Phi_s'$. Similarly for all the subsequent expressions in which ϕ_s' occurs. This shows that ϕ_s' is a stress of the kind contemplated.

The bearing of this on capillary phenomena will not be discussed here, because this is foreign to the objects of the present paper. It was necessary in this paper to show the general results flowing from the existence of l_s.

It should, however, be remarked that this stress, though not affecting the mechanical action of the field on a body as a whole, would affect the strains of a body, and probably be sometimes comparable in this effect with the similar effects resulting from the dependence of l on strain.

115. In conclusion, let me remark that in several respects the above investigations might be generalised. It is not hard to take account of the slipping of surfaces over one another, both with regard to reversible and irreversible phenomena. It is harder, but not impossible, to take account of the existence and relative motion in identically the same portion of space of two media, such as the ether and air, or as two different kinds of matter, as in diffusion and chemical phenomena (though, of course, in the two last cases, the two media do not probably really exist in the same portion of space —a statement not proven). I have refrained from this in the present paper for two reasons: first, to keep the length of the paper within reasonable bounds, and secondly, not to render the subject more intricate than is absolutely necessary. My aim has been not so much to establish incontrovertible results as to develope a new method of treatment, more powerful, and in reality much simpler than those which are in use to-day. If I have succeeded in convincing my readers that this method is worthy of study, the main object of the present paper is attained. Meanwhile the matters that have been just indicated can be left over for future consideration.